BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

A Historical and Critical Review of Mathematical Science

EDITED BY

EARLE R. HEDRICK
DAVID R. CURTISS WILLIAM R. LONGLEY

WITH THE ASSISTANCE OF

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J. H. M. WEDDERBURN

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1. Typewrite words and the very simplest formulas only.
2. *Do not* try to typewrite any complex formulas. Write them.
3. Keep a copy, and send the editors two copies, if you can.
4. *Do not* underline any symbols or any formulas.
5. Underline theorems with blue pencil (avoid ink).
6. Follow our recent styles in abbreviations, footnotes, etc.
7. Write carefully the (often misunderstood) capitals C K P S V W X Z.
8. Write ϵ, not ε. Write very carefully $\gamma\ \eta\ \kappa\ \lambda\ \nu\ \tau\ \upsilon\ \chi\ \omega$.
9. Among Greek capitals, use only $\Gamma\ \Delta\ \Theta\ \Lambda\ \Xi\ \Pi\ \Sigma\ \Phi\ \Psi\ \Omega$.
10. Punctuate carefully, especially in formulas; thus: 1, 2, $\cdots$, n.
11. Use the solidus (/) to avoid fractions in solid lines.
12. Use fractional exponents to avoid root signs everywhere.
13. Use extra symbols to avoid complicated exponents.
14. In typewritten formulas, l means "one"; to indicate "ell" in formulas, backspace and overprint /; thus: l. Similarly, O means "zero"; to indicate "cap O," backspace and overprint period; thus: O.
15. Avoid a dash over a letter, except for those shown below.
16. Some samples of unusual types available on monotype machines follow. A more complete list of all such types will be sent on request.

Light Face Greek—$\alpha\ \beta\ \gamma\ \cdots$ (all) $\mathrm{A\ B\ \Gamma\ \cdots}$ (all).

★ Light Greek Superiors—$^{\Delta}$ and $^{\alpha\ \beta\ \gamma\ \cdots}$ (all except ^{o}).

★ Light Greek Inferiors—$_{\Delta\,\Lambda\,\Sigma\,\Omega}$ and $_{\alpha\ \beta\ \gamma\ \cdots}$ (all except $_{o}$).

* Boldface Greek—$\boldsymbol{\alpha\ \beta\ \gamma\ \delta\ \varepsilon\ \zeta\ \eta\ \theta\ \kappa\ \lambda\ \mu\ \nu\ \xi\ \pi\ \varrho\ \sigma\ \omega}$ and $\boldsymbol{\Omega}$.

* Lightface German—$\mathfrak{a\ b\ c\ d\ e\ f\ p\ q}$ $\mathfrak{A\ B\ C\ D\ E\ F\ G\ H\ J\ K\ L\ M\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$.

* Boldface German—$\mathfrak{b}$ $\mathfrak{A\ B\ D}$.

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* Hebrew— א ב ג ד troublesome to handle.

★ Dashed Italics—$\overline{A}\ \overline{a}\ \overline{B}\ \overline{b}\ \overline{C}\ \overline{c}\ \overline{E}\ \overline{e}\ \overline{F}\ \overline{f}\ \overline{G}\ \overline{g}\ \overline{H}\ \overline{l}\ \overline{i}\ \overline{j}\ \overline{K}\ \overline{k}\ \overline{M}\ \overline{m}\ \overline{N}\ \overline{n}\ \overline{o}\ \overline{P}\ \overline{p}\ \overline{q}\ \overline{R}\ \overline{r}\ \overline{s}\ \overline{t}\ \overline{u}\ \overline{v}\ \overline{X}\ \overline{x}\ \overline{Y}\ \overline{y}\ \overline{Z}\ \overline{z}$.

★ Tilda Italics—$\tilde{A}\ \tilde{a}\ \tilde{e}\ \tilde{N}\ \tilde{n}\ \tilde{O}\ \tilde{r}\ \tilde{u}\ \tilde{y}$.

★ Tilda Greek—$\tilde{\alpha}\ \tilde{\epsilon}\ \tilde{\eta}\ \tilde{\omega}\ \tilde{\varpi}$.

★ Dashed Greek—$\overline{\alpha}\ \overline{\beta}\ \overline{\gamma}\ \overline{\delta}\ \overline{\eta}\ \overline{\theta}\ \overline{\mu}\ \overline{\nu}\ \overline{\rho}\ \overline{\omega}\ \overline{\Gamma}$.

★ Dotted Italic—$\dot{a}\ \ddot{a}\ \dot{e}\ \ddot{e}\ \ddot{g}\ \ddot{\imath}\ \dot{m}\ \dot{n}\ \dot{q}\ \dot{r}\ \ddot{r}\ \dot{u}\ \ddot{u}\ \dot{x}\ \ddot{x}\ \dot{y}\ \ddot{y}\ \dot{z}\ \ddot{z}$.

★ Dotted Greek—$\dot{\eta}\ \ddot{\eta}\ \dot{\theta}\ \ddot{\theta}\ \dot{\xi}\ \dot{\psi}\ \ddot{\psi}\ \ddot{\omega}$ (single dotted $\zeta\ \phi\ \delta\ \beta\ \gamma$; double dotted γ readily available).

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THE OCTOBER MEETING IN NEW YORK

The two hundred sixty-third regular meeting of the American Mathematical Society was held at Columbia University, on Saturday, October 27, 1928, extending through the usual morning and afternoon sessions. The attendance included the following sixty-seven members of the Society:

R. G. Archibald, A. A. Bennett, Benton, B. L. Brown, Carrié, Dadourian, Darkow, Dehn, Demos, Douglas, Dresden, Eisenhart, Feld, Fiske, Fite, Fort, Gehman, Gill, Gronwall, L. S. Hill, Hille, Hofmann, Hollcroft, Huber, Dunham Jackson, M. I. Johnson, R. A. Johnson, Kaplan, Kenny, Kholodovsky, Kline, Koopman, Mark Kormes, Lefschetz, McGiffert, Marden, Meder, Richard Morris, Mullins, C. A. Nelson, K. E. O'Brien, Ore, Pell-Wheeler, Pfeiffer, Pierpont, Post, Raudenbush, Ritt, Rutledge, Schoonmaker, Seely, Serghiesco, Siceloff, Simons, Smail, P. A. Smith, J. M. Thomas, Trjitzinsky, Weida, Weisner, M. E. Wells, D. E. Whitford, Whittemore, Wiener, W. A. Wilson, Margaret M. Young, Zippin.

At the meeting of the Council, the following persons were elected to membership in the Society:

Mr. Andrew Campbell Berry, Harvard University;
Miss Olive Margaret Hughes, Bryn Mawr College;
Mr. William Thomas Reid, University of Texas;
Professor Allen A. Shaw, University of Arizona;
Dr. Warren Jennison Willis, patent attorney, New York City.

The following members of the London Mathematical Society have entered the American Mathematical Society under the reciprocity agreement since the last meeting:

Mr. Francesco Tavani, London;
Professor T. P. Trivedi, Karachi, India.

Associate Secretary Dresden reported the following elections by mail vote of the Council:

To sustaining membership: Members of the Department of Mathematics, University of Wisconsin;

To ordinary membership:

Dr. Frederick Richard Bamforth, University of Chicago;
Professor David Francis Barrow, University of Georgia;
Mr. Leon Battig, Oberlin College;
Dr. Clifford Bell, University of California at Los Angeles;
Professor Arthur H. Blue, Western Union College;
Mr. Arthur Barton Brown, Harvard University;
Professor Helen Calkins, Sweet Briar College;
Mr. Paul T. Copp, Purdue University;
Mr. Paul Cramer, Purdue University;
Professor Louis Antoine Victor De Cleene, St. Norbert College;
Dr. Miltiades Stavros Demos, Harvard University;
Professor Carl Christian Engberg, University of Nebraska;
Mr. William Irvin Foster, Rochester Junior College;
Mr. Harold Sinclair Grant, University of Pennsylvania;
Professor Samuel Oliver Grimm, Lebanon Valley College;
Captain Elmer Ellsworth Hagler, Jr., United States Army;
Professor Marie Mathilda Johnson, Oberlin College;
Dr. Pierce Waddell Ketchum, University of Illinois;
Mr. Edward August Knobelauch, University of Pennsylvania;
Mr. Trueman Lester Koehler, Lehigh University;
Mr. Semen Arsenijevitch Lepeshkin, Brown University;
Professor Anna Marm, Bethany College;
Sister Mary Bertrand (Walton), Marywood College;
Mr. John Ellsworth Merrill, Case School of Applied Science;
Professor Elmer Beneken Mode, Boston University;
Mr. Joseph Kimbark Peterson, Harvard University;
Mr. Thurman Stewart Peterson, Ohio State University;
Miss Mina S. Rees, Hunter College;
Professor Charles Edward Schroeder, Boston College;
Professor Charles Louis Searey, University of Nevada;
Mr. James Singer, Princeton University;
Professor Carlton W. Smith, State Teachers College, Superior, Wis.;
Professor Atherton Hall Sprague, Amherst College;
Mr. Carl Walther Strom, Luther College;
Professor Emory Earl Walden, Lambuth College;
Professor Charles Ernest Weatherburn, Canterbury University College, Christchurch, New Zealand;
Professor Thomas Payne West, University of Idaho, Southern Branch;
Miss Jean Winston, University of Cincinnati;

Nominees of the Penn Mutual Life Insurance Company, Philadelphia: Messrs. J. E. G. Bryant, M. L. Johnson, Adolph F. Schwartz, Lawrence L. Stevens, Clarence E. Willows.

The Council accepted with thanks the invitation of Lehigh University to hold the Eastern Christmas meeting of 1929 in Bethlehem, Pa.

The Council accepted the recommendation of the com-

mittee on colloquia that Professor S. Lefschetz be invited to deliver the colloquium lectures at the summer meeting in 1930 at Brown University.

It was voted to approve the use of the name of the Society in the announcements of the American Yearbook; and the President was authorized and requested to appoint, each time for a period of three years, a representative of the Society on the Editorial Board of the Yearbook.

Professor Bennett presided at the morning session, and Professor Jackson at the afternoon session.

Titles and abstracts of the papers read at this session follow below. The papers of Franklin, Garver, Kasner, Stone, Suschkewitsch, Whyburn, and Zippin were read by title. Dr. Arnold was introduced by Associate Secretary Arnold Dresden, Mr. Rashevsky by Professor J. I. Taylor, and Professor Suschkewitsch by Professor H. H. Mitchell.

1. Professor T. R. Hollcroft: *Invariant postulation.*

The invariant postulation of a manifold, simple or multiple, on a variety in i dimensions is the number of invariants among the coefficients of the variety that are necessary and sufficient for the variety to contain a manifold of given nature. The invariant postulation of a certain manifold is obtained readily from the ordinary postulation in which both the nature and position of the manifold are taken into account. Invariant postulation, although a concept not heretofore used except in the case of points in the plane, is of importance mainly because of the geometric relations revealed by it. Some of these relations have been proved before by other methods, and some are new.

2. Dr. H. E. Arnold: *The rational space quintic curve of the second species and its relation to the rational plane quartic curve.*

The rational space quintic curve of the second species, R_3^5 (II), possesses ∞^1 quadrisecant lines, which are given by a pencil of binary quartics. The first part of the present paper concerns itself with the determination of conditions for special quadrisecants, obtained, in general, in terms of the Morley invariants. (See R. M. Winger, American Journal, vol. 36.) By means of these conditions a correspondence is set up between certain types of R_3^5 (II) and those of R_2^4 (the rational plane quartic curve). The R_3^5 (II) for which I_2 vanishes is determined by a binary quintic form whose roots represent coplanar points. In the second part of the paper several facts concerning this plane are obtained. Finally, some properties of R_3^5 (II) are derived from the Jonquière quartic, and a geometric meaning is given to the unique sextic apolar to the pencil of line sections of R_2^4 which pass through a fixed point of the plane.

3. Dr. Louis Weisner: *Invariants of a plane 5-point.*

From the complete system of invariants of a binary quintic, consisting of I_4, I_8, I_{12}, I_{18}, expressed in terms of the roots, invariants J_6, J_{12}, J_{18}, A_{27} of the associated plane 5-point are found by the method of complementary determinants explained by Coble. But these four invariants of the plane 5-point do not form a complete system because A_{27} turns out to be an alternating function. In the present paper a complete system is found to consist of J_6, J_{12}, J_{18}, J_{33}, the subscripts denoting the degrees of the invariants. The results are applied to a redetermination of the invariants of E. H. Moore's cross-ratio group of order 120.

4. Professor R. G. Archibald: *An arithmetical function analogous to Euler's ϕ-function.*

An arithmetical function $\omega(n)$ is defined as the number of integers relatively prime to n in the sequence $1\cdot 2\cdot 3/6,\ 2\cdot 3\cdot 4/6,\ \cdots,\ n(n+1)\cdot(n+2)/6$. E. Lucas (*Théorie des Nombres*, vol. I, 1891, p. 403) and R. D. Carmichael (*The Theory of Numbers*, 1914, p. 36) suggested the problem of finding the number of terms in the foregoing sequence which are relatively prime to n. As far as the author knows, a formula giving the number of such terms has not yet been published. In the present paper, by the use of Lucas' ψ-function, an explicit formula for $\omega(n)$ is obtained in terms of n if, where m is an integer relatively prime to 6, n is different from $2^k3m (k \geqq 0)$. In these cases, solutions of a set of linear congruences (always solvable) are to be obtained. If $\omega_s(30\ m)$ denotes the number of terms in the given sequence relatively prime to 6 m when $n = 30\ m$, and if m is any positive integer relatively prime to 6, $\omega(36\ m) = 2\omega(18\ m) = \omega(6\ m) + \omega_s(30\ m) = \omega(12\ m) + \omega(24\ m)$. Moreover, if m_1 and m_2 are two relatively prime numbers each relatively prime to 6, and if $k \geqq 2$ and $l \geqq 1$, $\omega(m_1m_2) = \omega(m_1)\omega(m_2)$, $\omega(2^km) = \omega(2^k)\omega(m)$, $\omega(3^km) = \omega(3^k)\omega(m)$, $\omega(2^l3^km) = \omega(2^l3^k)\omega(m) = \omega(2^l)\omega(3^km) = \omega(3^k)\omega(2^lm)$.

5. Dr. Jesse Douglas (National Research Fellow): *On the inverse problem of the calculus of variations.*

The problem is as follows: given a system of paths, under what conditions can it be identified with the totality of extremals of a calculus of variations problem? We represent the system of paths by the differential equations $d^2x^i/dt^2 = H_2^i(x,\ p)$, where $p^i = dx^i/dt$ (Annals of Mathematics, vol. 29 (1928), pp. 143–168), and the calculus of variations problem by $\int F(x,\ p)dt = \text{minimum}$. H_2^i is homogeneous of the second degree and F of the first degree in the variables p. Necessary and sufficient conditions are found to be represented by the system of covariant partial differential equations (1) $F_{|i,j} - F_{|j,i} = 0$, where the stroke denotes $\partial/\partial p$ and the comma covariant differentiation based on the affine connection $\Gamma^i_{j,k} = \frac{1}{2}\partial^3 H_2^1/\partial p^j \partial p^k$. There are three cases as to (1): (a) it may have no solutions F (the general case); (b) it may have a class of solutions depending linearly on a finite number of arbitrary constants; (c) it may have a class of solutions involving

arbitrary functions. In the present paper, sufficient conditions are established for the case (b). The only operations required for the application of these conditions are (covariant) differentiation and the algebraic discussion of a certain finite system of linear equations.

6. Dr. Jesse Douglas: *A generalization of homogeneous functions and of Euler's equations.*

The results here are a by-product of the writer's work on spaces of K-spreads (see this Bulletin, Jan.-Feb., 1928). Let $H^{\lambda\mu\cdots\nu}_{\alpha\beta\cdots\gamma}$, where there are r lower and s upper indices each varying from 1 to K, be a system of functions of the variables p^i_α; $i=1, 2, \cdots, N$; $\alpha=1, 2, \cdots, K$; we imagine the system p^i_α to be distributed into N sets of K variables, with i fixed for each set. Let these sets undergo cogredient linear transformation: $q^i_\alpha = A^\beta_\alpha p^i_\beta$ reciprocally $p^i_\alpha = B^\beta_\alpha q^i_\beta$, the coefficients A^β_α, B^β_α being therefore the same for every value of i. Suppose that, identically,

$$(1) \qquad H^{\lambda\mu\cdots\nu}_{\alpha\beta\cdots\gamma}(q) = A^\rho_\alpha A^\sigma_\beta \cdots A^\tau_\gamma B^\lambda_\phi B^\mu_\psi \cdots B^\nu_\omega H^{\phi\psi\cdots\omega}_{\lambda\mu\cdots\nu}(p).$$

The property expressed by this equation reduces to homogeneity of degree $r-s$ in case $K=1$. Generalizations of Euler's equations of the first and higher orders are established. Generalization is further made to the case where (1) involves as factor in its second member a power of the determinant $|A^\beta_\alpha|$ (relative tensor); also to the case where the independent variables form N systems ${}^ip^{\lambda\mu\cdots\nu}_{\alpha\beta\cdots\gamma}$, which undergo cogredient tensor transformation.

7. Mr. Nicholas Rashevsky: *An electrostatic problem.*

The solution of the electrostatic problem for a space limited by two infinite parallel planes, one of which is provided with a set of equidistant parallel blades, at right angles to the planes, is given in the form of an infinite product. The use of the first factor of the product as the first approximation leads to elliptic integrals. By a logarithmic transformation, the above solution gives also the solution of the corresponding problem for two coaxial cylinders, one of which is provided with equidistant radial blades. The solution is applied to the calculation of voltage factors in certain special types of three-electrode tubes.

8. Dr. Lulu Hofmann: *Remarks on a certain aspect of plane projective transformations.*

In two projectively related planes, the metric properties of corresponding segments on corresponding lines and corresponding angles at corresponding points are known. This paper investigates the metric properties of corresponding segments radiating out from corresponding points. The principal result is as follows. With every fixed pair of corresponding points P_0, P_0' in two projective planes π, π', and every value of a positive quantity k, is associated a pair of elliptic quartics Q_k, Q_k', corresponding under the projectivity, which are the loci of all pairs of corresponding points P, P', such that the distances of any two such points from the

respective fixed points have the same ratio: $\overline{PP}_0 = k\overline{P'P}_0'$. These quartics $Q_k[Q_k']$ have one double point at the infinite point of the vanishing line of $\pi[\pi']$ and the other at the fixed points $P_0[P_0']$. They consist of two odd branches lying on different sides of the respective vanishing lines and having the equi-segmental axes for asymptotes. They are tangent to the circles about $P_0[P_0']$ as centers at the cyclic points. In particular for $k=1$, $Q_1[Q_1']$ represents in $\pi[\pi']$ the locus of the double points determined by the projectivity as $\pi'[\pi]$ is successively laid on $\pi[\pi']$ in the ∞^1 ways possible such that P_0 and P_0' coincide.

9. Professor Edward Kasner: *Polygenic functions of two complex variables.*

In a paper read at the Bologna Congress, the author began the study of polygenic functions of two variables by showing that the partial derivatives are represented by a pair of uniformly parametrized circles or *clocks*. (For the case of one variable, see Science, Dec. 16, 1927.) He now studies the three partial derivatives of second order, which give rise to quite complicated configurations of interest. (For the case of one variable, see the Transactions of this Society, Oct., 1928.)

10. Professor Raymond Garver: *A solution of the quartic equation.*

By the use of two theorems in the author's paper recently published in the Messenger of Mathematics, the quartic equation is solved in a manner that seems to present some interest. The present paper will appear in the American Mathematical Monthly.

11. Professor M. H. Stone: *Hermitian symmetric operators in abstract Hilbert space.*

The present paper resolves the characteristic value problem for an operator T with the following properties: (a) T orders to each element of a set $\mathfrak{F}$ everywhere dense in an abstract complex Hilbert space $\mathfrak{H}$ an element of $\mathfrak{H}$; (b) if f and g are elements of $\mathfrak{F}$, $Q(f, Tg) = Q(Tf, g)$, where Q is a Hermitian bilinear form determining the metric of $\mathfrak{H}$; (c) if two elements f and f^* exist in $\mathfrak{H}$ such that, for every g in $\mathfrak{F}$, $Q(f, Tg) = Q(f^*, g)$, then f belongs to $\mathfrak{F}$. The theory includes the known theory of operators for which $\mathfrak{F} \equiv \mathfrak{H}$, developed by Hilbert and others, and leads to analogous results. It includes also the results obtained by J. von Neumann for "real" operators, as stated by him in the Göttinger Nachrichten, 1927, Heft 1, pp. 1–57.

12. Mr. P. M. Swingle: *End sets of bounded continua irreducible between two points.*

A study is made of continua, irreducible between two points, based particularly upon various types of their subsets called "end sets." Among other results, necessary and sufficient conditions for the indecomposability of such continua are obtained.

13. Professor Philip Franklin: *The operators of quantum mechanics.*

In this note we study some properties of the transition from the Hamiltonian function to the Schrödinger operator. In particular we show that any attempt to define the correspondence by linear operators must fail to yield a result independent of coordinates.

14. Professor Philip Franklin: *Algebraic matric equations.*

The equations here studied are of the form

$$P(X) = A,$$

where the left member is a polynomial, and A and X are matrices. For the special case

$$X^n = A,$$

we determine a general form for X giving all the roots. With some restrictions on X, the results are extended to the more general case. In particular, after further specializing X to be expressible as a polynomial in A, we obtain from our results those of Roth (Transactions of this Society, vol. 30, p. 579).

15. Professor Anton Suschkewitsch: *On a generalization of the associative law.*

In this paper it is pointed out that the theorem of Lagrange to the effect that the order of a group is divisible by the order of a subgroup may be proved without the complete use of the associative law. A generalization of this law is proposed, and some theorems regarding it are established.

16. Professor G. T. Whyburn: *Local separating points of continua.*

The point P of a continuum M will be called a *local separating point* of M if there exists a compact open set R containing P and such that if C is the component of $M \cdot \overline{R}$ which contains P, then $M \cdot \overline{R} - P$ is separated between some two points of $C - P$. The point P of an E-set M is called an *indispensable point* of M if $M - P$ is not an E-set. In this paper a study is made of the local separating points of continua M lying in any locally compact, metric, and separable space. These points are characterized in various ways, some in terms of the indispensable points of certain cuttings of M, and theorems are obtained elucidating their properties. Perhaps the most interesting ones are the following: (1) all save possibly a countable number of the local separating points of M are points of order two of M; (2) every uncountable set of local separating points of M contains an uncountable subset every pair of points of which cuts M. The notion of a *local separating continuum* is introduced, and many important properties of local separating points are shown to hold for local separating continua.

17. Professor Edward Kasner: *Higher partial derivatives of polygenic functions.*

The higher partial derivatives of polygenic functions depend upon the curves of approach and also upon the correspondence between the curves. The order of differentiation must be taken into account. Thus in general

$$\frac{\partial^2 w}{\partial z_1 \partial z_2} \neq \frac{\partial^2 w}{\partial z_2 \partial z_1}$$

In fact equality, for all approaches, holds only when w is an analytic function. Thus there are in general four distinct derivatives of second order. For rectilinear approach these reduce to three. Higher orders are also discussed.

18. Professor Norbert Wiener: *The summability of Fourier series.*

The author applies his new method in Tauberian theorems to the demonstration of the Hardy-Littlewood theorems on the convergence and summability of Fourier series.

19. Mr. Leo Zippin: *The Janiszewski-Mullikin theorem.*

The author proves that in a one-dimensional continuous curve containing at least one simple closed curve the Janiszewski-Mullikin theorem is not satisfied. In an acyclic continuous curve it is vacuously satisfied. For the proof, he requires a theorem of Gehman's on subsets of a plane continuous curve which lie on an acyclic subcontinuous curve which he generalizes to n dimensions.

20. Professor W. A. Wilson: *Certain problems related to the cutting of a simply connected plane region by a continuum.*

Let R be a simply connected plane region with a bounded frontier F and C be a bounded continuum contained in $\overline{R}$ such that $C \cdot F$ is the sum of two closed sets α and β between which C is irreducible and C is neither indecomposable nor the union of two indecomposable continua. A study is made of the frontiers of the regions into which C divides R for various types of frontiers F. When C is a simple arc and F is a regular frontier, the nature of the frontiers depends upon whether F contains 0, 1, or 2 continua irreducible between α and β, and the various possibilities are catalogued. If C is an arc, F is the irreducible sum of two continua H and K neither of which disconnects F, and the ends of C lie on different components of $H \cdot K$, the frontiers are $C+H$ and $C+K$. Various sufficient conditions for the extension of this last result to the case that C is not a simple arc are given. In this connection it is also shown that, if F is the union of two continua H and K and is disconnected by neither of them precisely two components of $H \cdot K$ are accessible by a continuum from R and can therefore be joined by an irreducible continuum contained in $\overline{R}$ and connected im kleinen at all of its points save possibly those on $H \cdot K$.

21. Professor W. A. Wilson: *Some properties of multicoherent continua.*

This paper appears in full in the present issue of this Bulletin.

22. Professor J. F. Ritt: *On the zeros of exponential polynomials.*

By an exponential polynomial is meant a function $a_0e^{\alpha_0 z}+\cdots+a_ne^{\alpha_n z}$, with constant a's and α's. The distribution of the zeros of such functions has been studied by Tamarkin and by Pólya. In the present paper it is proved that if the quotient of two exponential polynomials is an integral function, then the quotient is an exponential polynomial.

23. Professor B. O. Koopman: *On the singularities of linear partial differential equations.*

We consider the system of m equations of any order in m unknown functions, in the independent variables $(x, \cdots, x_n)$, and having analytic coefficients. A slight modification of the Cauchy-Kovalevsky calculus of limits establishes the following: If R_{2n} is a $2n$-dimensional region of the complex $x_1 \cdots x_n$-space containing no singular point of the system, and S_{2n-2} a regular portion of the analytic surface

$$S(x_1, \cdots, x_n) = 0,$$

in R_{2n}, not tangent to a characteristic, there exists a region P_{2n} of R_{2n}, containing S_{2n-2}, throughout which every solution of the Cauchy problem is analytic, whatever be the (analytic) Cauchy data on S_{2n-2}. From this it follows that if

$$f(x_1, \cdots, x_n) = 0$$

is an isolated analytic singularity of an analytic solution of the equations, and if it contains only ordinary points, it is a characteristic. These results have been obtained in the case of two independent variables by LeRoux and Delassus by methods which do not admit of extension; and by Delassus and Hadamard for n variables under very restrictive assumptions.

24. Dr. Morris Marden (National Research Fellow): *Zero-free regions of a certain partial fractions.*

This paper tries to find simple regions of the plane which do not contain any zeros of the partial fraction

$$\sum_{i=1}^{n} \alpha_i/(z-a_i),$$

when α_i is a complex constant. It presents a few such regions where the points a_i are collinear, or where the numbers a_i are all real or appear in conjugate imaginary pairs, as well as in cases where the points a_i are arbitrarily distributed in the plane.

25. Dr. T. C. Benton: *On continuous curves which are homogeneous except for a finite number of points.*

A continuous curve is said to be homogeneous if for every pair of points x and y belonging to the curve there exists a (1, 1) continuous correspondence of the *curve* into itself which makes x correspond to y. This paper classifies the plane continuous curves which are homogeneous except for a finite number of points. For one non-homogeneous point c, the set consists of a finite ($\neq 2$) or countably infinite set of simple closed curves which have only c in common. For two such points, the set consists of a finite number ($\neq 2$) of arcs joining the two points. For three or more points a skeleton set is formed by replacing by a single arc the set of arcs which join two non-homogeneous points but contain no other non-homogeneous points. The skeleton set may be either a set of arcs having only one point in common or a simple closed curve, or a set which is in (1, 1) correspondence with the projection of one of the regular polyhedra or of one of the semi-regular polyhedra on one of its faces in such a way that none of the projections of edges have any intersections except projections of vertices. The case of unbounded plane homogeneous continuous curves is also taken up and it is found that except for the one-point case where the set corresponds to a finite number of rays leaving the non-homogeneous point, there is no such set other than the whole plane.

ARNOLD DRESDEN,
Associate Secretary

THE FIFTY-SIXTH REGULAR MEETING OF THE SAN FRANCISCO SECTION

The fifty-sixth regular meeting of the San Francisco Section was held at the University of California on Saturday, October 27, 1928. The Chairman, Professor E. T. Bell, presided. The total attendance was 37, including the following 23 members of the Society:

Alderton, A. D. B. Andrews, Basoco, E. T. Bell, Bernstein, Buck, Cajori, Corbin, Garver, W. L. Hart, M. W. Haskell, E. R. Hedrick, Hotelling, Frank Irwin, Jerbert, D. N. Lehmer, S. H. Levy, W. A. Manning, W. E. Milne, Noble, T. M. Putnam, A. R. Williams, B. C. Wong.

The following officers were elected for the coming year:

Chairman, Professor M. W. Haskell; Secretary, Professor B. A. Bernstein; Program Committee, Professors E. T. Bell, H. F. Blichfeldt, R. M. Winger, and B. A. Bernstein (ex officio).

It was resolved that the members of the Section living in the Northwest be requested to change the place of the next Summer meeting from the Oregon Agricultural College to the University of California, in order that the Section may participate in the meetings of the Pacific Division of the American Association for the Advancement of Science to be held in June.

It was resolved unanimously: That the Society be requested to make the meetings of the Section Meetings of the Society.

It was decided to hold the next Fall Meeting at the University of California on October 19, 1929.

Titles and abstracts of the papers presented at the meeting follow below. Professor Bell's first paper was given at the request of the Program Committee. The fourth and fifth papers of Professor Bell, Professor Garver's second and third papers, and Professor Whyburn's paper were read by title.

1. Professor E. T. Bell: *The theory of algebraic numbers in the light of Kronecker's program.*

It is suggested as at least plausible, and as a unique piece of scientific irony, that the present unrest concerning the foundations of analysis which was inaugurated by Kronecker half a century ago, reacts destructively upon his own theory of algebraic magnitudes in which it originated.

2. Professor E. T. Bell: *Invariant sequences of polynomials.*

There are determined all sequences of uniform functions of one variable such that the derivative of each term of a given sequence is equal to the preceding term, and each term is changed by a linear transformation of the variable into a multiple of itself, the multiplier being a function of the rank of the term alone, and the linear transformation and the multiplier function being the same for all terms. The terms are necessarily polynomials. By linear transformations on the rank and the variable, any number of distinct sequences having the stated properties for any given linear transformations on their variables, can be replaced by new sequences, all of which are transformed alike by the same linear transformation of the variable, and hence are instances of a single sequence of the original kind. In this respect the theory of any number of distinct sequences of the kind described can be unified. The number of invariant sequences of polynomials is infinite, to the same degree as the set of all uniform functions of one variable.

3. Professor E. T. Bell: *On certain finitely solvable equations between arithmetical functions.*

The number of equations between arithmetical functions for which we can prove either that they have no solution, or only a finite number of solutions, and in the latter case state the solutions, is negligible. Fourteen new equations of this type, involving the binary quadratic class number, are completely solved or are shown to be insolvable. As a point of interest, the solution is achieved by utilizing the conjecture of Descartes in 1638, which was proved only in 1911 by Dubois, (see Dickson's *History*, vol. II, pp. 276, 302).

4. Professor E. T. Bell: *Outline of a theory of arithmetical functions in their algebraic aspects.*

In the Journal of the Indian Mathematical Society for October, 1927, Dr. Vaidyanathswamy calls attention to the existence of inverses of multiplicative arithmetical functions. He remarks that they do not seem to have been utilized before, as they are not explicitly mentioned in Dickson's *History*. The reference in the History which covers them implicitly is vol. I, p. 323, item 174. The present paper is a restatement in more general form of a complete algebraic theory of arithmetical functions, not necessarily multiplicative, which has been constructed by the writer in several papers during the past sixteen years. This restatement generalizes and simplifies all of the preceding work.

5. Professor E. T. Bell: *An interpretation of certain decomposable algebraic forms as functions of divisors.*

The forms are in n indeterminates, where n is an arbitrary constant integer and the coefficients of the n linear factors into which the forms are decomposable are in any given algebraic number field. The decomposition is double, in that it has a dual interpretation, first when the indeterminates are elements of an abstract field, second, when they are elements of the irregular field of all extended numerical functions. When the relevant algebraic number field is defined by a primitive root of unity, the forms degenerate to circulants in each interpretation; another degeneration is that to the resultant of any pair of algebraic equations.

6. Professor Florian Cajori: *Early determinations of the heights of mountains.*

The author makes a detailed examination of the processes and instruments most probably used by the Greeks in determining the heights of mountains. He sets forth the modifications introduced in the Middle Ages and the Renaissance, as well as the improvements in geodetic procedure and in the construction of instruments during the seventeenth and eighteenth centuries. He deals next with the early development of the barometric method, also of the hypsometric and pendulum methods.

7. Professor Florian Cajori: *Hobbes, Wallis, and Barrow on principles of mathematics.*

The author points our that Wallis founded mathematics on arithmetic, that Hobbes and Barrow founded it on geometry. With Wallis arithmetic was an abstract science, to which geometry was subordinate. According to Hobbes, Wallis mistook the study of symbols for the study of geometry. According to Barrow, numbers themselves could not even be added to or subtracted from one another. The discussions carried on involved also the definitions and postulates of Euclid, the consideration of angles of contact and the concept of limits. In particular, Barrow and Wallis defended Euclid's definition of proportion against the attacks of Hobbes who pronounced the definition unworkable because it demanded an infinite number of trials. Hobbes criticises Wallis's reasoning by induction. Wallis anticipated some nineteenth century ideas on arithmetization.

8. Professor Florian Cajori: *Isaac Newton's edition of Varen's geography.*

The author finds that Newton made numerous corrections in Varens *Geographia Generalis* and added thirty geometric figures. Of particular interest is Newton's correction of a numerical error in Varen's account of the measurement of a degree of the earth's meridian due to Snell, and the use of Snell's value in correcting Varen's table for the distance a mountain of known height can be seen at sea (disregarding refraction). These corrections show conclusively that in 1672 Newton was familiar with the result of Snell's measurement. On no other occasion,

previous to writing the *Principia*, is it known definitely what meridian value Newton actually used.

9. Professor Raymond Garver: *Quartic equations with certain groups.*

A general method, employing the Tschirnhaus transformation, is devised which may be of some value in the problem of determining all equations of a certain degree with a prescribed group. The method can be applied easily to quartic equations; the results are equivalent to those given by Seidelmann in volume 78 of the Mathematische Annalen.

10. Professor Raymond Garver: *Linear fractional trans formations on quartic equations.*

The reduction of quartic equations to normal forms by means of linear fractional transformations is usually carried out with the aid of either algebraic geometry or invariant theory. This paper considers the problem from a purely algebraic point of view, and treats a number of normal forms.

11. Professor Raymond Garver: *Two notes on cyclic cubics.*

This paper gives (1) a simple proof that every cyclic cubic is of Seidelmann's form $x^3-3(p^2+3q^2)x+2p(p^2+3q^2)=0$, and (2) a Tschirnhaus transformation that serves to reduce every such cubic to the normal form $y^3-3(P^2+3)y+2P(P^2+3)=0$.

12. Dr. A. R. Jerbert: *Quintuples of curves in four-space.* Preliminary report.

A system of five differential equations of the form $dx_i/dt=a_{ij}x_j(i, j=1, 2, 3, 4, 5)$ serves as analytic basis. A fundamental system of solutions is interpreted as defining, to within a projectivity, 5 points in 4-space. As the parameter t varies, each of the five points describes a curve. The resulting configuration consists, therefore, of five curves in one-to-one correspondence. The projective properties of the configuration are readily expressed in terms of the invariants and covariants of the defining system.

13. Professor W. M. Whyburn: *On the distribution of the characteristic values of differential and integro-differential systems.*

The general system of linear differential equations $\sum_{j=1}^{n}\sum_{s=0}^{k}{}_sP_{ij}(x)y_j^{(s)}=0$, $(i=1, 2, \cdots n)$, where the functions ${}_sP_{ij}(x)$ are summable on X: $a\leqq x\leqq b$, is studied. The matrix calculus is used to develop adjoint relations and the results are shown to coincide with the classical results for the special cases of a single equation of the kth order $(i=j=1)$ and a system of n equations of the first order $(k=1)$. A canonical form is derived for the system when the determinant of the matrix $({}_kP_{ij}(x))$, k fixed, is different from zero on X.

The above differential system is considered when one of the coefficients ${}_rP_{mt}$, depends upon a parameter λ in such a way that ${}_rP_{mt}(x, \lambda)$ is sum-

mable on X for each fixed λ on L: $L_1<\lambda<L_2$, continuous in λ on L uniformly with respect to x on X, and bounded numerically on XL by a summable function of x. Necessary and sufficient conditions that there exist continua of characteristic values of the differential system and a set of homogeneous boundary conditions are derived. As a corollary of this result it is shown that a self-adjoint system cannot have a continuum of characteristic values and furthermore, the values of λ for which such a system is incompatible form intervals that are everywhere dense on L. Corresponding systems of integro-differential equations are considered.

14. Professor W. A. Manning: *The class of triply and quadruply transitive groups.*

It is known that if a triply transitive group of degree n and class $u(>3)$ contains a permutation of even order on u letters, $n\leqq 2u$; and that for quadruply transitive groups of class $u(>3)$ in which some permutation of degree u is of even order, $n\leqq 2u-1$. The author now announces that if one of the permutations of degree u of a triply transitive group of class $u(>3)$ is of odd prime order p, $n<2pu/(p-1)+3$. For all quadruply transitive groups of class $u(>3)$, $n\leqq 2u+1$. This last result betters Bochert's limit by unity.

15. Mr. M. A. Basoco: *On the Fourier expansions of doubly periodic functions of the third kind.*

The doubly periodic functions of the third kind may be classified into two groups according as the excess m of the number of zeros over the number of poles of the function is positive or negative. In a series of papers in the Annales de l'Ecole Normale, (1884–1888), Appell has developed a theory for obtaining the Fourier expansion of these functions. It is found that the theory is applicable in a practical manner to functions for which $m<0$, while for functions such that $m>0$, the theory, while complete from a function theoretic point of view, does not lead, in general, to arithmetically useful results, since it leaves certain constants expressed in the form of definite integrals, the actual evaluation of which is quite impracticable.

Using other methods, numerous expansions of functions exhibited as quotients of theta products with $m>0$ have been obtained in a form suitable for use in the theory of numbers.

B. A. BERNSTEIN,
Secretary of the Section.

THE AMHERST COLLOQUIUM

A short program, five lectures on one subject instead of the usual two series by different speakers, proved nevertheless decidedly attractive. The list of those registering for Professor A. B. Coble's Colloquium numbered seventy-seven, nearly all being present on Tuesday, September 4th, at the opening lecture. The college opened the large lecture-room in Appleton Hall, with ample blackboard space and excellent acoustic properties. Professor H. L. Rietz, Vice-president of the Society, presided and introduced the lecturer. As it is expected that the lectures will be published promptly in the Society's Colloquium Series, no full account of the contents is here necessary. Excerpts from the quite full syllabus and from a supplementary sketch kindly furnished by Dr. Coble will suggest the extensive résumé and new researches, which sum up a line of the lecturer's publications extending back at least to 1915. Other authors cited were principally Stahl, Schottky, Picard, Cayley, Frobenius, and the Encyklopädie.

The title of the series was *The determination of the tritangent planes of the space sextic of genus four*. The standard definition of a function theta of p variables is given, by the aid of a square array which determines its (quasi) periods. The zero points of the variables are shifted by half-periods, leading to definitions of $2^{2p}-1$ odd and even thetas and their characteristics. This is ground-work for the proper subject of the lectures, as follows.

If the original periods are replaced by new periods (integer combinations of the old and vice versa) the $2^{2p}-1$ half periods are permuted, and the 2^{2p} odd and even functions also, according to a finite group G_{NC}. A geometric representation of this group is found by taking the half periods as points in a finite space S_{2p-1}, modulo 2. The collineation

group G_{NC} has then an invariant null system N and the 2^{2p} theta characteristics behave under it like quadrics whose polar systems coincide with N. The finite geometry indicates a *basis notation* in which the points (or half periods) and quadrics (or odd and even thetas) are expressed by means of $2p+2$ subscripts (Annals, 1916, p. 101).....

Products and powers of these first-order thetas aid in defining thetas of the second and higher orders, whose relations form a bridge to the algebraic and abelian functions. Not these, however, but the groups concerned and their invariants were the central object, and a variety of geometric pictures were invoked to make the theory vivid.

The geometric theory of the theta functions of genus two is in the main related to figures determined by six points. We use the three six-points: P_6^1, six points on a line; P_6^2, six points in a plane (usually on a conic); and P_6^3, six points in space on a norm cubic curve C^3. The P_6^1 and P_6^3 are *associated sets* of points,..... The P_6^2, if on a conic, is self-associated; otherwise it determines projectively an associated Q_6^2.

With this geometric background there is a projective invariant theory.....A first object is to show that the theta-relations are satisfied by virtue of the projective relations among the invariants and covariants. The invariant relations and the modular relations ($\breve{u}=0$) are identical; the same is true of the covariant relations and the theta-relations ($\breve{u}\neq 0$). A second object is to show that the period transformations of the functions appear in the geometry as the passage from one set P_6 to a congruent set P_6' under Cremona transformation.....

The functions of genus three are intimately connected with the figure of seven points P_7^2 of the plane, and the eight base points P_8^3 of a net of quadrics in space. These eight base points P_8^3 are congruent under Cremona transformation to 36 such 8-points and give rise to a Cremona group of order $8!\cdot 36$. Also the 8-point can be expressed parametrically in terms of modular functions, and period

transformations (mod 2) produce the same Cremona group. In this case both modes of approach are unified by the actual existence of a curve of genus three which defines the functions: namely, the locus of nodes of the quadrics of the net. Similarly in the case $p=4$ the set of 10 nodes of a Cayley symmetroid are congruent to $2^8 \cdot 51$ such 10-points and give rise to a Cremona group of order $2^8 \cdot 51 \cdot 10!$. Also Schottky has found that these ten nodes can be expressed parametrically in terms of modular functions of genus four. Again the period transformations (mod 2) produce the Cremona group. Thus far however a curve of genus four which defines the functions has not been discovered. The isolation of the tritangent planes of the normal space sextic curve of genus four is then dependent upon the detection of a curve of genus four attached to a symmetroid which will degenerate to a curve of genus three if either two nodes of the symmetroid coincide or four nodes are coplanar.

With two sessions on each of the first two days, the course was finished on Thursday. On Tuesday afternoon many of the participants motored to Mount Holyoke College, where the local mathematical faculty served tea and escorted the visitors to points of interest. Wednesday afternoon, Smith College colleagues were similarly hospitable. For Amherst, Professors Olds and Esty, with President Pease, were indefatigable in arrangements and personal courtesies; and the reception by President and Mrs. Pease at the Lord Jeffery Inn Tuesday evening created a nucleus for the memories of natural beauty, serene intellectuality, and pervasive friendliness which formed the setting of this twelfth Colloquium.

H. S. WHITE

THE HEROIC AGE OF GEOMETRY*

BY J. L. COOLIDGE

The remarks which I shall have the honor to make today are addressed primarily, if not exclusively, to the older generation. The prophet Joel describes the golden age to come by saying "Your old men shall dream dreams, and your young men shall see visions." That distinction is eternally valid. Youth has visions of the future. Age dreams of the past. Our younger members are facing forward, looking towards the splendid mathematical discoveries that will be made during the coming decades, among which will be included those which they themselves will have made. It stirs the blood and fortifies the courage to feel that one is called upon to contribute in this way to the advancement of science, and that a grateful posterity will recognize what one has done. If the present passion for materialism continues, the number of contributions to mathematical science in the next hundred years may not greatly exceed half a million. Go on, oh younger generation, happy in the thought that the mathematicians of a century hence will, by an unfailing instinct, pick out your own particular contributions from all the rest!

For you who, like myself, belong to the generation that is passing, who have reached the age to dream dreams, I propose a humbler task. I propose that we look backward. The most notable epoch in all the long history of geometry, the heroic age, was almost exactly a hundred years ago. It was the moment when geometric supremacy was passing from France to Germany, a fact which the French geometers were slow to recognize, for Chasles, in his *Aperçu historique des méthodes en géométrie*, regretted that he could not

* An address read by invitation of the program committee before the Society, and the Mathematical Association of America, December 28, 1928.

speak of the work of the German geometers because he could not read their language. I shall try to show you some of the problems which interested the geometers of that time, and the methods chosen for their solution. I shall try to point out what was most significant in their work, and what was their influence on the progress of science.

Karl Wilhelm Feuerbach was born in 1800, contemporary with Macaulay and von Moltke, and died in 1834, professor at the gymnasium in Erlangen. In 1822, he published in Nürnberg a pamphlet entitled *Eigenschaften einiger merkwurdigen Punkte des geradlinigen Dreiecks*. The original of this, which may have been his doctor's dissertation, is hard to find, but a reprint appeared in 1908. The writer is concerned with the geometry of the triangle. When he finds two notable points he determines their distance, when there are three he determines the center and radius of the circumscribed circle. The methods are entirely elementary. The inscribed and escribed circles seem to attract him particularly, and finally he proves, quite casually, on page 48, that the circle through the feet of the altitudes passes through the middle points of the sides, and is tangent to the inscribed and escribed circles. This is the first known proof of the nine-point or Feuerbach circle theorem, though the author himself missed three of the nine points. Previous writers had found the center and radius of the circle, Bevan giving their determination as a problem in 1804.* The existence of nine points thereon was discovered in 1821 by Brianchon and Poncelet.† It is no disparagement to Feuerbach that he did not see Bevan's article which appeared in an obscure English journal, while the work of the French writers may well have appeared after his own had gone to press.

The fate of his theorem was that of many another;‡

* Leybourn's Mathematical Repository, vol. 1, p. 18.

† Gergonne's Annales de Mathématiques, vol. 11, p. 215.

‡ See Macaulay, *History of the nine-point circle*, Proceedings of the Edinburgh Mathematical Society, vol. 11 (1892).

first it was overlooked, then rediscovered. Steiner announced it without proof in 1828* and found the remaining three points in 1833.† Next Terquem proved it completely in 1842.‡ Salmon proved the tangency part in 1860 as did Casey,§ although they called the circle merely a six-point circle. Since that time there has been a perfect cloudburst of demonstrations. It has been said that the study of the philosophy of Kant has risen almost to the rank of one of the liberal professions; the same is measurably true of proving Feuerbach's theorem. The last adept would seem to be Sawayama, who gave nine demonstrations, all presumably new, in 1911.‖ Perhaps there is an element of absurdity in all this, but can we deny that we owe to Feuerbach the most beautiful theorem in elementary geometry that has been discovered since the time of Euclid?

Steiner's important contributions to elementary geometry during the period that we are considering were by no means limited to proving the Feuerbach theorem. He brought out and developed several of the most important principles connected with the modern geometry of the circle, notably¶ the idea of the power of a point with regard to a circle, the centers of similitude and radical axes. These ideas, though perhaps new to Steiner were not strictly speaking new to geometry, as they had already been presented by Gaultier,** Durande†† and Poncelet.‡‡ But Steiner handled his methods with consummate skill; witness his beautiful solution of Malfatti's problem to construct three circles, each tangent to two sides of a triangle, and to the other two circles, which he gave

* See his Collected Works, Berlin, 1881, pp. 195, 196.

† Ibid., p. 491, Note.

‡ Nouvelles Annales de Mathématiques, vol. 1, p. 197.

§ Quarterly Journal of Mathematics, vol. 4, p. 153.

‖ L'Enseignement Mathématique, vol. 13 (1911).

¶ *Einige geometrische Betrachtungen*, Journal für Mathematik, vol. 1 (1826).

** Journal de l'École Polytecnique, vol. 9 (1813).

†† Annales de Mathématiques, vol. 11 (1820).

‡‡ *Propriétés Projectives des Figures*, Paris, 1822.

without proof, and for which no simple proof was published till that of Hart thirty years later.* Moreover, his proof of the Feuerbach theorem was contained in a paper which was important in another way, for it deals with constructions possible with the aid of a ruler and one circle fully drawn and with given center. This is not, of course, the first attempt to replace the Platonic instruments, ruler and compass. There were, for a long time, two opposing tendencies, one to extend the list of permissible instruments, the other to restrict it. In the latter category we might mention the idea of using a compass with fixed opening, which goes back to Abul Wafa in the 10th century† or the really astonishing work of Mascheroni in proving that every determination of a point that is possible with ruler and compass, is also possible with the compass alone.‡ The idea of using a ruler and given circle was first broached by Poncelet,§ but was carried through independently and in far greater detail by Steiner.

While in France and Germany distinguished mathematicians were doing their best to build up the ancient science of geometry, further East others were equally zealous in what must have seemed uncommonly like an effort to tear it down. The period we are considering included the birth of the non-euclidean geometry, at present a perfectly respectable branch of mathematics, but a heretical doctrine one hundred years ago. The "fons et origo" was skepticism about Euclid's twelfth postulate, that dealing with parallels. Candid persons had felt for a long time that this was not really as self evident as a reputable axiom ought to be, and countless attempts had been made to prove it. Now one of the classical methods of procedure in mathematics is the reductio ad absurdum, and quite naturally it occurred to those who were trying to prove this axiom, to see what would

* Quarterly Journal of Mathematics, vol. 1 (1855).
† See Tropfke, Journal Asiatique, (5), vol. 5 (1855), p. 226.
‡ *Geometria del Compaso*, 1797.
§ *Propriétés Projectives des Figures*, 1822, pp. 187–189.

happen if they replaced it by some other assumption about parallels. This was first tried by Saccheri at the middle of the 18th century, and many others after him. In each case things went well until the writer slid into some unwarrantable assumption in order to show that he was in difficulties when he wasn't. It took six decades from the time of Saccheri, and the genius of Gauss, to see that the parallel axiom was really independent of the others. But Gauss was busy with other matters and did not take the trouble to publish his results, so that the credit for first publicly announcing the non-euclidean geometry goes to a Russian, Lobachevski, who, on February 12, 1826, read before the Physico-Mathematical Society of Kasan a paper entitled *Exposition succincte des principes de la géométrie.** No trace of the manuscript of this address has ever been found, but we know from Lobachevski's later writings what must have been included therein. It is also permissible to doubt that it deserved the adjective succinct. His Russian article on the *Principles of geometry*, which appeared in the Kasanskij Wjestnik, the Kasan News in 1829–30, purports to be an extract from the previous paper, and covers 66 pages. It is very incomplete, and we only get to the root of Lobachevski's ideas by studying his subsequent publications.

Let us hasten to bracket with Lobachevski, the independent discoverer Johann Bolyai, whose *Appendix, Scientiam Absolute Veram Spatii Exhibens* was attached to his father Wolfgang's *Tentamen Juventutem Studiosam in Elementa Matheseos . . . . Introducendi*, which was published at Maros Vasarhely in 1832.

The truly startling thing about these two works is their similarity. It is not to be wondered at that, when Bolyai first saw Lobachevski's work in 1835, he made the natural mistake of believing that it was copied from his own.† We have learnt in a hundred years that there are a good many

* For an account of Lobachevski, see Engel, *Lobatschefskij*, Leipzig, 1898.

† See Stäckel, *Wolfgang and Johann Bolyai*, Leipzig, 1913, pp. 140 ff.

different ways of setting to work to establish the non-euclidean geometries. Kipling has written somewhere:

"There are nine and sixty ways, of constructing tribal lays,
And every single one of them is right."

It is much the same in constructing non-euclidean geometries. It is therefore really astonishing, that Lobachevski and Bolyai showed the following similarities:

1. Both defined parallel lines as the limiting positions of intersecting ones.

2. Both reached at a bound the hyperbolic geometry, overlooking completely the possibility of an elliptic geometry, where there were no parallels.

3. Both introduced at an early stage the horosphere, which is the surface orthogonal to a bundle of parallel lines, bringing out the remarkable fact that on this surface we have the euclidean geometry.

4. Both pointed out that the formulas of spherical trigonometry could be established without the parallel axiom.

5. Both studied the ratio of the non-parallel sides of a quadrilateral, two of whose sides are equal and parallel.

Of course there are divergences as well as similarities in the writings of the two. Lobachevski made continual use of the parallel angle associated with a given distance, that is, the angle which the parallel to a given line, through a point at a given distance from that line, makes with the normal. Bolyai manipulated skillfully his theorem that the ratio of the sines of two angles of a triangle is that of the circumferences on the opposite sides as diameters. Lobachevski is more prolix, and especially interested in trigonometric formulas, Bolyai is brief, but he brings out the function of the space constant in admirable fashion. My own final impression is that the points of similarity are far more remarkable than those of difference.

It was many years before the significance of non-euclidean geometries was fully understood, and developed beyond the point reached by the discoverers. It was only in 1854 that the third classical geometry, the elliptic geometry of Riemann

was first exhibited.* Skepticism lingered; in order to convince the average man, or the average mathematician, it was necessary to see the non-euclidean geometry at work. A sphere, or rather a hemisphere where opposite points of the equator count as identical, gives a very good elliptic plane; a beautiful specimen of a Lobachevskian plane, or rather of a part of one, was found by Beltrami in a surface of constant negative curvature.† These examples did not go beyond two dimensions. Klein seized on the theory of projective measurement, casually thrown out by Cayley‡ to give a perfect example of Lobachevskian or Riemannian geometry in as many dimensions as may be desired. Lie's theory of continuous groups threw a new and valuable light on the questions involved; contributions came in from various quarters. Since the beginning of the present century the subject has been somewhat transcended, owing to the newer methods of differential geometry, and the theory of relativity. Of the various methods of attack, that of Riemann based on the study of quadratic differential forms has shown itself best able to meet the new demands. But we owe an enormous debt to Lobachevski and Bolyai, not only for enriching geometry, but for initiating a movement which has been of incalculable importance to philosophy. Our whole modern conception of mathematics as a logical system based on arbitrary axioms may be traced back to their pioneer work.

We have had occasion already to speak of Steiner once or twice; we have not yet said a word about his most important work, *Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander*,§ one of the corner stones on

* *Ueber die Hypothesen welche der Geometrie zur Grunde liegen.* See his Collected Works. 2d ed., Leipzig, 1892, pp. 272 ff.

† *Saggio di interpretazione della geometria non-euclidea*, Giornale di Matematiche, vol. 6 (1868).

‡ Klein, *Ueber die sogenannte Nicht-Euklidische Geometrie*, Mathematische Annalen, vol. 4 (1871); Cayley, *Sixth Memoir on Quantics*, Philosophical Transactions, vol. 149 (1859).

§ Berlin, 1832. See his Collected Works, vol. 1.

which the whole subject of synthetic projective geometry rests. Who were the predecessors, from whom, presumably, Steiner drew inspiration? They were certainly French. Among the earlier writers there were, of course, Pascal, Desargues and Brianchon. Then Carnot, although preceding the modern movement, exercised considerable influence through his *Géométrie de Position*,* especially through his theory of transversals. More important by far was Poncelet, whose *Propriétés Projectives des Figures*, composed in a Russian prison in 1813, first saw the light in 1822. Poncelet's root idea may be stated about as follows. A geometric figure may be wonderfully simplified by a central projection from plane to plane, or by a homology in plane or in space. What are the properties of figures which are invariant under such a transformation, how can theorems about the simplified figures be stated in invariant terms so as to give properties of the original figures? We see at once that the essentials consist in concurrence, collinearity, coplanarity and cross ratios. Poncelet also introduced a rather cumbrous system of what he called "ideal chords" determined not by the given curve, but by another, to get rid of imaginary intersections. Finally he laid stress on the method of polar reciprocation with regard to a conic or quadric to establish the principle of duality. This last principle was a good deal talked about at the time; a long and not very edifying dispute between Gergonne and Poncelet as to priority of discovery is spread over the pages of the Annales de Mathématiques.† The former conceived duality in a broader spirit as an inherent characteristic of geometric figures, but did not follow this important idea very far; the latter did not much believe in any duality he did not see, so he created it by polarization.

All of these threads were drawn together by Steiner in his development of projective geometry, which is nothing if not "systematisch." He starts with a point, the points

* Paris, 1803.

† Volume 18 (1827–28).

of a line, and those of a plane. The plane is the simplest two-dimensional locus of points, the point the simplest two-dimensional envelop of planes, the line is the simplest one-dimensional figure of points or planes, the pencil of lines is a self-dual figure. The principle of duality appears at the very start, and a good part of the work is printed in double columns. Cross ratios of points, lines and planes are then defined, and it is shown how they are unaltered by projections and intersections. Steiner talks a good deal about "projective forms" but the definition is rather mixed with theorems about them. The underlying thought is that they are one-dimensional forms whose elements are in one-to-one correspondence, with equality of corresponding cross ratios. Steiner shows easily enough that two forms which are connected by a finite number of projections and intersections are projective in this sense; his proof of the converse is faulty owing to his inability to handle the question of the continuity of the projective relation. Starting with these data, Steiner obtains all the other figures he wants to study by construction; a conic is given by the intersections of corresponding lines in two projective pencils, and enveloped by lines connecting corresponding points in two projective ranges.

Every modern student of projective geometry will see that the fundamental ideas of his science are here set out in order. Steiner laid a very solid foundation whereon his successors might build. It is true that the Latin school of geometers, including such mathematicians as Chasles and Cremona, drew their inspiration more from Poncelet than from Steiner, but the latter had a notable lineal descendant in von Staudt. This profound thinker perceived two flaws in the perfection of the Steinerian structure. In the first place, although cross ratios are projectively invariant, their original definition is metrical; secondly, there is no satisfactory treatment of imaginary elements. These defects he set out to remedy in heroic fashion.* Each set of four ele-

* See his *Geometrie der Lage*, 1847, and *Beiträge zur Geometrie der Lage*, 1856–1860.

ments of a fundamental one-dimensional form he calls a "throw" and associates with a number; four harmonic elements, in right order, are associated with the number -1. He then defines the sum and the difference of two throws, and shows how a throw can be found corresponding to every rational number. At that point he was blocked as he had no means of handling irrational throws, no Dedekind cut. The lack was later filled by Klein.* Von Staudt's other highly original idea was to define an imaginary point as an elliptic involution of points, to which a sense of description has been attached. He showed how projective geometry, enlarged by these new elements, followed the old laws and permitted the old constructions. His culmination is the treatment of a complex throw.

The synthetic geometry of Poncelet and Steiner was vigorously pursued for a century, the last heroic figure being that of Reye. In recent years we have tended to question the wisdom of too sharp a separation between synthetic and analytic methods, and to confine our researches either to tinkering with the fundamental assumptions, or to the field of projective differential geometry. But the synthetic methods have a compelling charm when rightly presented, and afford a most admirable training for every geometer, wherever his specialty may lie.

The great progress of synthetic geometry in the years we have been considering was, fortunately, not at the expense of analytic geometry; on the contrary, the rivalry between the two was of great value to mathematical science. The progress of algebraic methods kept pace with that of the synthetic ones; strangely enough the first writer we must mention in this new connection is our previous acquaintance, Karl Wilhelm Feuerbach. In 1827 we find him publishing his *Grundriss zur analytischen Untersuchung der dreieckigen Pyramide.* His object here is analogous to that previously pursued in the study of the triangle, to establish relations

* Mathematische Annalen, vol. 7.

between the distances between the notable points of the pyramid. But he now passes from synthetic to analytic methods, and consequently arrives at very different theorems from those reached before. To begin with, and most important, he sets up a novel set of coordinates. He has the idea of linear dependence firmly fixed in his mind. Suppose that the coordinates of a variable point are expressed as linear combinations of those of four fixed non-coplanar points

$$X = ax + bx' + cx'' + dx''',$$
$$Y = ay + by' + cy'' + dy''',$$
$$Z = az + bz' + cz'' + dz''',$$
$$1 = a + b + c + d.$$

The quantities a, b, c, d connected by the last identity may be taken as a new system of coordinates. They may be interpreted as the distances to the four faces of a tetraedron, with right algebraic signs, divided by the corresponding altitudes. A linear equation among them will give a plane. Feuerbach has an additional tool in an identity which he attributes to Lagrange, but which I am ashamed to say I have not been able to find in his writings. If $A, B, C, D,$ and E be five points in space, their distances are connected by the relation

$$\begin{vmatrix} 0 & AB^2 & AC^2 & AD^2 & AE^2 & 1 \\ BA^2 & 0 & BC^2 & BD^2 & BE^2 & 1 \\ CA^2 & CB^2 & 0 & CD^2 & CE^2 & 1 \\ DA^2 & DB^2 & DC^2 & 0 & DE^2 & 1 \\ EA^2 & EB^2 & EC^2 & ED^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

He proves a good many theorems with the aid of his new tools; some are interesting, others unimportant. Perhaps his most notable result is to give in very simple form the equation of a sphere tangent to four given spheres. It must not be forgotten that Feuerbach never heard of a determinant.

If we must give Feuerbach full credit for the independent discovery of one form of tetrahedral coordinates, we must hasten to point out that he was not the only discoverer. A more important mathematician than he made the same discovery in the same year, August Ferdinand Möbius.* The idea of this able geometer was to base geometric theorems on the properties of the center of gravity of a system of points. If two points be given, any point of their line will be the center of gravity of the two, provided they be endowed with proper masses, positive or negative. These masses are taken as homogeneous coordinates for the point. Two masses whose sum is zero are made to correspond to the infinite point of the line. A similar definition, when three non-collinear points are given, will give the homogeneous coordinates of a point in their place while four non-coplanar points will yield the homogeneous barycentric coordinates of a point in space, exactly proportional to the coordinates of Feuerbach. In representing a straight line, Möbius uses linear dependence on two points thereof, while a plane is given by linear dependence on three. This parametric method yields, of course, the simplest means of handling linear problems. When it comes to studying a curve, in plano or in spatio, Möbius represents the coordinates of a point thereof as rational functions of a given parameter, while two parameters are used for a surface. As the coordinates used are homogeneous, he replaces rational functions by polynomials, whose degree gives the degree of the curve or surface. He knew too much to imagine that all algebraic curves and surfaces could be expressed in this way, but he confined himself to rational varieties, and in particular to quadratic ones. A notable feature is that he has a clear grasp on the fundamental idea of an invariant, that is to say, a property of a geometric figure which is unaltered by the transformations of a certain group. He studies successively motions which keep distances unaltered, affine transformations where

* *Der barycentrische Calcul*, Leipzig, 1827.

areas or volumes are altered in a fixed ratio, and collineations which he defines as transformations which carry collinear or coplanar points into collinear or coplanar ones. His fundamental relations here are the cross ratios of four collinear points, triangle ratios involving six coplanar ones, and tetrahedral ratios involving eight non-coplanar ones. If four points of a plane, no three collinear, be invariant for a collineation, all points of their net of rationality are invariant. Möbius assumes that the transformation must therefore be the identity. He expresses a collineation by a linear transformation of his homogeneous barycentric coordinates; he grasps the principle of duality both in the form of Gergonne, and that of Poncelet.

It would be a mistake to imagine that Möbius' contributions to geometry were limited to the publications of the *Barycentrischer Calcul*, and contemporary articles dealing with the same subject. We are indebted to him for the discovery of the null system in space, and a geometric theory of circle transformations in the plane which is useful in the study of the simplest functions of a complex variable. The form of the *Barycentrischer Calcul* is not exactly that which we should choose today, but the wealth of fruitful ideas is remarkable.

It would certainly seem that Feuerbach and Möbius were enough to share the credit for discovering tetrahedral or trilinear coordinates; but such is not the case. Omitting independent discoveries of a somewhat later date, we must now render full credit to Julius Plücker for his work of 1828.* He started out with the deliberate intention of showing that all of the beautiful results which Poncelet and Steiner had reached by synthetic methods were easily obtained by algebraic analysis. His interest in trilinear coordinates was much less than that of the previous writers mentioned; they appeared to him merely as a sort of abridged notation. Thus if A and A' be the expressions for the distances of a point

* *Analytisch-geometrische Entwickelungen*, vol. 1, 1828; vol. 2, 1832.

from two given lines, the equations $A=0$, $A'=0$ and $A+\nu A'=0$ will represent respectively the first line, the second line or an arbitrary line through their intersection. An almost identical point of view was contemporarily, though doubtless independently, expressed by Bobilier,* to whom we owe also the equation of the first polar of a point with regard to a given curve.† This idea of using a single letter to replace a whole polynomial gave Plücker an easy method of proving not only straight line theorems, but a number of beautiful properties of circles that had been found synthetically by the French geometers. When it comes to conic sections, he uses Cartesian coordinates, in his first volume, giving a good deal of attention to change of axes, and polar reciprocation.

Between writing the first and the second volume, Plücker became very much impressed with the idea of duality, so that Volume 2 is entirely given to line geometry in a plane. The coefficients of a line are the ratios of the coefficients in its equation when expressed in terms of homogeneous Cartesian point coordinates. A large part of the book is given to curves of the second class. The domain is real, so that an ellipse can be defined as a curve of the second class which has a tangent pointing in every direction. The volume, like its pre ecessor, ends with polar reciprocation.

One has the impression, on the whole, that Plücker's work at this moment was less original than that of Möbius, though easier to read, and yielding more results. It is also to be remembered that his greatest contributions to algebraic geometry came later. We owe to him the coordinates of a line in space, which, in the perfected form devised by Klein, are highly important in that branch of science vaguely called "Higher Geometry." We are also indebted to Plücker for the beautiful identities connecting the numbers of point and line singularities of an algebraic plane curve. Let us mention in passing, that we do not surely know to this day

* Annales de Mathématiques, vol. 18 (1827–28).

† Ibid., vol. 19 (1828–29), p. 108.

when a set of solutions of these equations in positive integers necessarily corresponds to an existing curve.

It would certainly seem that with the names of such geometers as Poncelet, Steiner, Möbius, and Plücker, we might close the list of those who contributed to geometric science in the second period just one hundred years ago; it would seem so, and we should be justified, were it not for the the fact that there lived in the small but important town of Göttingen, a man by the name of Johann Carl Friedrich Gauss who published his *Disquisitiones Generales circa Superficies Curvas* in 1827.* This work is not only the firm basis on which all the mass of subsequently discovered differential geometry rests, but is also the most important individual contribution ever made to that branch of mathematical science.

What were the most novel and fruitful ideas contained in Gauss' geometric work? To begin with, and most important, Gauss was the first writer to make consistent use of the method of parametric representation of a surface, that is to say, the method of expressing the Cartesian coordinates of the points of a surface as functions of two independent variables. The idea was not absolutely new. Euler used it in approaching the general problem of the applicability of one surface to another, his exact words being:†

"Et quia per naturam superficiarum quaelibet coordinata debet esse functio binarum variabilium a se invicem non pendentes." Unfortunately this important idea did not bear immediate fruit. There was much interest in differential geometry at the beginning of the nineteenth century, especially on the part of the French mathematicians. Writers like Monge and Dupin who made really important contributions, regularly expressed z as a function of x and y, Cauchy equated a function of the three variables to zero.

* Commentationes Societatis regiae Gottingensis, vol. 6 (1828); vol his Collected Works, vol. 4, Göttingen, 1873. An English translation with notes by Thompson was published in Princeton, N. J., 1902.

† Leonardi Euleri Opera Posthuma, vol. 1, St. Petersburg, 1862, p. 494.

The superiority of the Euler-Gauss method is so evident that it "jumps to the eyes." The modern developments of differential geometry would never have been possible without it. Gauss first states it in his *Theoria Attractionis* of 1813.* He uses it also in his prize memoir presented to the Scientific Society of Copenhagen in 1822,† but it is in the *Disquisitiones* that it is first convincingly set forth.

A second important feature of Gauss' treatment is the use made of spherical representation. Gauss considers not one surface but two. Previous writers paid attention to the direction cosines of the normals; Gauss treats these as the Cartesian coordinates of a point on an auxiliary sphere. The beauty of the conception can be seen in the following way. In the plane we may associate with each curve a circle of unit radius whose points correspond to the oriented normals of unit length of the given curve. The curvature is the ratio of the lengths of corresponding infinitesimal arcs of circle and curve. In the same way the total curvature of a surface is the ratio of corresponding infinitesimal surface elements on sphere and surface. It was in this fashion that Gauss presented his idea; it had been given in substance, if not in so many words, by Olinde Rodrigues,‡ who wrote the equation

$$\frac{1}{RR'} = \left(\frac{\partial X}{\partial x}\right)\left(\frac{\partial Y}{\partial y}\right) - \left(\frac{\partial X}{\partial y}\right)\left(\frac{\partial Y}{\partial x}\right).$$

Still the credit for actual statement is Gauss', the year 1816.§

The important part of Gauss' work on the measure of curvature is not in the statement given, but in the demonstration that this expression is invariant under every transformation of the surface which leaves distances invariant, so that mutually applicable surfaces have, at corresponding points, the same measure of curvature. Moreover, this

* Collected Works, vol. 5, 1867, p. 14.
† Ibid., vol. 4, pp. 189 ff.
‡ Correspondence de l'École Royale Polytechnique, vol. 3, 1816, p. 168.
§ Works, vol. 8 (1900), p. 367.

particular invariant, and its extension to higher spaces, has proved to be fundamental in non-euclidean geometry, as pointed out by Riemann and Beltrami and many others.

A good proportion of the *Disquisitiones* is devoted to the study of geodesic or shortest lines. Gauss reaches the differential equations for such lines by straightforward methods of the calculus of variations. His equations show at once the characteristic property that the principal normal to the curve is normal to the surface, a fact first discovered by Johann Bernoulli.* What is rather curious is that Gauss does not point out that geodesic lines may be characterized by the fact that geodesic curvature is everywhere zero, even though Gauss himself appears to have been the first to study this sort of curvature.† He presently proves the existence of geodesically parallel curves. The latter part of the essay is largely devoted to the study of geodesic triangles.

If the ultimate influence of the *Disquisitiones* was incalculable, we can not affirm that its importance was immediately felt.‡ He had an immediate and devoted follower in Minding, who published an entirely Gaussian article on the development of curves on surfaces in 1830.§ But in 1831, we find, Mlle. de St. Germain writing¶

"Si par rapport aux surfaces on avait besoin de connaître la mesure de courbure . . . le mémoire d'Euler contient tout ce que l'on sait d'important à cet égard." The surprising thing is not that a French woman should write this, but that Crelle's Journal should publish it.

It is fair to say that Mlle. de St. Germain will have nothing to do with any measure of curvature but mean curvature. It seems unlikely that the ideas of Gauss were understood or appreciated at all in France till the indefatigable

* See his letter to Leibnitz in Gott. Gul. Leibnitii et Johann Bernoullii Commercium, vol. 1, Lausanne, 1745, p. 393 near bottom.

† See his fragment on *Seitenkrummung*, Works, vol. 8, pp. 386 ff.

‡ In this matter I cannot agree with Stäckel, *Gauss als Geometer*, Gauss' Collected Works, vol. 10, Part II, Section IV.

§ Journal für Mathematik, vol. 6.

¶ Ibid., vol. 7.

Liouville lectured on them in the autumn of 1850.* Perhaps we may say that the final consecration only came in 1867 when Bonnet† proved that every set of solutions of the Codazzi equations determines a single surface completely, except for motions of space.

Such was the work of the leading geometers of one hundred years ago. What shall we say of them today? Well, for my part, I am willing to quote Scripture and say "There were giants in the land in those days." Think of Poncelet in a Russian military prison at Saratoff laying down the fundamental principles of projective geometry, Lobachevski and Bolyai breaking the chains of servitude to Euclid that had lasted for twenty-one centuries, Gauss laying an absolutely secure foundation for the geometry of any manifold in any space. Such work calls for ability of a high order. Moreover, there is one common element in all this work to which I should like to call your attention, the very nice balance between specific results and general theory. A distinguished Scots philosopher, whose lectures I once had the privilege of attending, was forever insisting that it was the great task of philosophy to "Combine the univairsal and the parteecular in a higherrr uneetie." It seems to me that geometry is called upon to do much the same thing. No individual theorem, be it that of Feuerbach, or even that of Pythagoras, can rise to the dignity of a general mathematical principle. Every mathematical proposition or system must be given the greatest possible breadth, in reason. But I wonder whether we are not today in some danger of extinguishing the vital spark in geometry by the excessive abstraction and generality of the results which we seek and publish. It is my personal credo that geometry is a branch of art. I find more emotional appeal in a Raphael Madonna or a Gothic cathedral than I could ever get from a picture of "Things in general" or a building intended for all possible purposes.

Perhaps the difficulty is inevitable, and there remain no

* See Liouville, Comptes Rendus, vol. 32 (1851), p. 533.

† Journal de l'École Polytechnique, vol. 25, Cahier 42, pp. 31 ff.

geometric results to be found which are at once so definite and yet so general as what has been found in the past. This may be so, but I like to think otherwise. I do not so far despair of the Republic. If anyone had asked Lagrange, who died but fifteen years before the beginning of the epoch we have considered, whether he believed there was much yet to do in geometry, he might well have given a hesitant answer. Let us have confidence that our five-thousand-year old science of Earth-measurement has still some beautiful secrets which she will yield when challenged by anyone who is sufficiently able and sufficiently fearless. Let us believe that

> "Some noble deed of note may yet be done
> Not ill-befitting men that strove with gods."

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THE CRITICAL POINTS OF FUNCTIONS AND THE CALCULUS OF VARIATIONS IN THE LARGE*

BY MARSTON MORSE

1. *Introduction.* The conventional object in a paper on the calculus of variations is the investigation of the conditions under which a maximum or minimum† of a given integral occurs. Writers have accordingly done little with extremal segments that have contained more than one point conjugate to a given point. An extended theory is needed for several reasons.

One reason is that in applying the calculus of variations to that very general class of dynamical systems or differential equations which may be put in the form of the Euler equations, it is by no means a minimum or a maximum that is always sought. For example, if in the problem of two bodies we make use of the corresponding Jacobi principle of least action the ellipses which thereby appear as extremals always have pairs of conjugate points on them, and do not accordingly give a minimum to the integral relative to neighboring closed curves, so that no example of periodic motion would be found by a search for a minimum of the Jacobi integral. In general if one is looking for extremals joining two points, or periodic extremals deformable into a given closed curve, the a priori expectation, as justified by the results of this paper, in general problems, would seem to be that many more solutions would fail to give a minimum than would give a minimum. Even if the ultimate object is

* An address presented before the Society at the request of the program committee, April 6, 1928.

† For work on the absolute minimum see Bolza, *Vorlesungen über Variationsrechnung*, 1909, pp. 419–437, and Tonelli, *Fondamenti di Calcolo delle Variazioni*, vol. 2. Further references will be found in these works.

narrowed down to the search for minima the existence of such minima is tied up in the large with the existence of extremals which do not furnish minima.

However, the real feeling of the author is that the classical calculus of variations is both logically and esthetically incomplete. The object of the present paper is to show by selected theorems and examples something of the more extended results now within reach of the science.* The author wishes to say, however, that in view of the tremendous technical difficulties involved, and the scope of the investigations now being carried on it is bound to be several years before a final idea of the possibilities of the new theory can be realized.

No theory in the large can well escape analysis situs. It is not to be expected, however, that pure analysis situs should contain all the results necessary for all the applications. For the present purposes a fundamental paper in analysis and analysis situs is the author's paper cited below.†

To reduce the study of the existence and nature of extremals to the study of critical points, admissible curves in any problem in the calculus of variations are replaced by nearby curves composed of a succession of short extremals.‡ The value of the given integral along the resulting broken extremal g is regarded as a function f of the coordinates of the end points of the component short extremals. Critical points of f occur when and only when g has no corners.

* Of the earlier work the most significant from the points of view of this paper is the "minimax principle." See Birkhoff, Colloquium Publications of this Society, vol. 9, *Dynamical Systems*, pp. 133–139.

† Marston Morse, *Relations between the critical points of a real function of n independent variables*, Transactions of this Society, vol. 27 (1925), pp. 345–396. This paper will be referred to as Morse I.

A *critical point* of a function is a point at which all of its first partial derivatives vanish.

‡ This idea is doubtless rather old. The manner in which it has been applied is variable. See Birkhoff, loc. cit. In proving his own theorems in his work not yet published the author has had occasion to go into many questions connected with the application of this idea that have never previously been considered.

Thus the search for extremals is reduced to the search for critical points of f.

Up to the present, a study has been made of three fields, first of extremals joining two fixed points, second of closed extremals, and third of extremals from a point to a manifold cut transversally by the manifold.

The first of these fields occupies the principal place in the classical calculus of variations. The second is of central importance for the theory of dynamical systems. The third leads to a geometry suggestive of real algebraic geometry without the algebra. Particularly interesting are the relations in the large that may be found between straight lines, hyperplanes, hyperspheres, and closed manifolds.

2. *The Hypotheses on the Integrand.** Let S be a closed region in the space of the variables $(x_1, \cdots, x_m) = (x)$. Let

$$F(x_1, \cdots, x_m, r_1, \cdots, r_m) = F(x,r) \tag{1}$$

be a positive, analytic function of its arguments, for (x) on S, and (r) any set not (0). Suppose further that F is positively homogeneous of order one in the variables (r). We employ the parametric form, taking $F(x, x')$ as our integrand, where (x') stands for the set of derivatives of (x) with respect to the parameter t. We suppose that the problem is *positively regular*, that is, that

$$\sum_{ij} F_{r_i r_j}(x,r)\eta_i\eta_j > 0, \quad (i,j = 1,2, \cdots, m), \tag{2}$$

* For that which concerns the classical theory of the calculus of variations in space the reader may refer to the following articles:

Mason and Bliss, *The properties of curves in space which minimize a definite integral*, Transactions of this Society, vol. 9 (1908), pp. 440–466.

Bliss, *The Weierstrass E-function for problems of the calculus of variations in space*, Transactions of this Society, vol. 15 (1914), pp. 369–378.

Bliss, *Jacobi's condition for problems of the calculus of variations in parametric form*, Transactions of this Society, vol. 17 (1916), pp. 195–206.

Hadamard, *Leçons sur le Calcul des Variations*, vol. 1, Paris, 1910.

Carathéodory, *Die Methode der geodätischen äquidistanten und das Problem von Lagrange*, Acta Mathematica, vol. 47 (1926), pp. 199–236.

for (x) and (r) as before, and (η) any set not (0) nor proportional to (r).

In general, for the theorems in the small in this paper, the hypothesis $F>0$ may be omitted, and the hypothesis that F be analytic replaced by the requirement that F be of class C'''. Further details will not be given here.

3. *Conjugate Points and their Orders.* Our future work depends not only upon a suitable definition of conjugate points but also of their orders.

Let there be given an extremal g which is an ordinary analytic curve joining a point P to a point Q. Let λ be the direction of g at P. Let the direction cosines of the directions neighboring λ be admissibly* represented as functions of $(m-1)$ parameters (α). Let s be the arc length of the extremal through P measured from P in the direction determined by (α). On the extremals issuing from P with the directions determined by (α) the coordinates (x) will be analytic functions of s and (α). Let $\Delta(s)$ be the jacobian of the coordinates (x) with respect to s and (α), evaluated for the parameters (α) that determine λ.

Each point on g, not P, at which $\Delta(s)$ vanishes, is called a conjugate point of P, and the order of the vanishing of $\Delta(s)$ the order of that conjugate point.

The order of a conjugate point is shown to be invariant† under admissible space transformations, and to be at most $m-1$. A conjugate point of the rth order will be counted as if it were r distinct conjugate points.

4. *The Type Number of an Extremal.* We shall first consider extremals joining two fixed points P and Q. Before turning to a theory of all such extremals it is necessary to consider one such extremal alone. Suppose g is an ordinary

* That is, the direction cosines are to be analytic functions of the parameters (α) of such sort that not all of the jacobians of $m-1$ of the direction cosines with respect to the parameters (α) are zero.

† Although conjugate points have been defined in various ways the notion of the order of a conjugate point does not seem to have received the attention it requires.

analytic curve without multiple points. The author has shown how g and a neighborhood of g can be mapped analytically, on a straight line segment h and a neighborhood of h. For present purposes then we can suppose that g is a segment of the x_1 axis.

Let us cut across g with n hyperplanes perpendicular to g. These hyperplanes divide g into $n+1$ successive segments. Suppose the hyperplanes are placed so near together that no one of these $n+1$ segments contains a conjugate point of its initial end point. Let us number these hyperplanes in the order of increasing x_1. Let P_i be any point near g on the ith one of these hyperplanes. Let the points

$$P,\ P_1,\ P_2, \cdots,\ P_n,\ Q$$

be successively joined by extremal segments neighboring g, and let the resulting broken extremal be denoted by E. Let $[u]$ be the set of the coordinates, other than x_1, of the points P_i. The value of the given integral J taken along E will be an analytic function of the variables $[u]$, and will be denoted by $J(u)$.

The function $J(u)$ will have a critical point when $[u]=[0]$. Let H be the quadratic form making up the terms of second order in $J(u)$ in a Taylor's expansion about $[u]=[0]$. We have the following fundamental theorem.*

If Q is a conjugate point of P of the rth order, the rank of H is $p-r$. If Q is not conjugate to P the rank of H is p, and its type number† is the sum of the orders of the conjugate points of P preceding Q, where $p=(m-1)n$.

Note particularly that the type number of H will be unchanged if we increase n to any arbitrary integer. The pre-

* For the case $m=2$, the proof of this theorem has already appeared See Marston Morse, *The foundations of a theory in the calculus of variations in the large*, Transactions of this Society, vol. 30 (1928), pp. 213–274. This paper will be referred to as Morse II.

† By the type number of H is meant the number of negative coefficients appearing in H when H has been reduced by a real, non-singular, linear transformation to squared terms only.

ceding theorem was developed primarily for use in the following theory in the large. Incidentally, it may be used to give what is probably the definitive answer as to the generalization of the famous Sturm separation theorem, for the case of systems of m ordinary, second order, self-adjoint, linear, differential equations in m dependent variables.

5. *Conjugate Points characterized by Deformations.** The entities deformed are families of curves joining P to Q and depending on μ parameters. They are called *μ-families*. The μ parameters are supposed represented by a point P on a closed μ-dimensional complex† C_μ. Each point P on C_μ corresponds to a curve of the family. The curves are supposed to vary continuously with the position of P on C_μ.

Let g be the given extremal, and let J_0 be the value of J along g. Suppose Q is not conjugate to P, but that there are k points ($k>0$) conjugate to P and preceding Q. We are going to consider continuous deformations of μ-families subject to the condition,

$$J \leqq J_0 - e^2, \qquad e > 0, \tag{3}$$

imposed on each member of the μ-family. We have the following characterization of the number of conjugate points in terms of deformations.

Corresponding to any sufficiently small neighborhood N of g, there exists within N an arbitrarily small neighborhood N_1 of g, and an arbitrarily small positive constant e, with the following properties. The type number k is the least integer k for which the $0, 1, \cdots, (k-2)$-families on N_1, satisfying (3), can be deformed into a single curve on N, subject to (3), and for which there exists at least one $(k-1)$-family which cannot be so deformed into a single curve.

* The theory of these deformations in the small has already been developed in the paper Morse II for the case $m=2$. The work appearing there for $m=2$ applies with hardly any change for a general m.

† Terms referring to pure analysis situs are used in the senses defined by Veblen in The Cambridge Colloquium, 1916, Part II, *Analysis Situs*.

A theorem similar to this but without use of the genera notion of μ-families is given by Birkhoff* for the case of closed trajectories of dynamics when $k=1$. For $k=1$ our theorem refers to a 0-family of curves joining P to Q, that is to a pair of curves joining P to Q. The theorem states that there exists a pair of such curves in N_1, which satisfy (3), but which cannot be deformed into each other subject to (3) without passing out of N.

A simple example of the theorem is obtained by considering the integral of the arc length of an n-sphere. Let the extremal g be an arc of a great circle slightly longer than a semi-circle. If $n=2$, then $k=1$, and the statement of the preceding paragraph is clearly true. For a general n, $k=n-1$, and one can apply the theorem as stated.

If $k=0$, the theorem says nothing. However, the theorem of the calculus of variations commonly known as Osgood's Theorem† is seen to fill the gap in a most natural way.

6. *Theorems in the Large.* The preceding theorem is a typical deformation theorem in the small. One can also consider the mutual relations of μ-families in the large. In particular one can introduce the notion of "bounding" or "non-bounding" families. It is by means of such considerations that one arrives at the following theorems in the large.

Suppose the region S homeomorphic with the interior and boundary of an $(m-1)$-sphere, and that S's boundary is *extremal convex*. That is, suppose there exists a positive constant e so small that any extremal segment on which $J<e^2$ and which joins two boundary points will lie interior to S except at most for its end points.

If P_0 and Q_0 are any two points of S there exist in their respective neighborhoods points P and Q which are joined by no extremals on which P is conjugate to Q.

Such a pair of points will be termed *non-specialized*. For such a pair there will be at most a finite set of extremals

* Birkhoff, *Dynamical systems with two degrees of freedom*, Transactions of this Society, vol. 18 (1917), p. 249.

† Osgood, Transactions of this Society, vol. 2 (1901), p. 273.

joining P to Q with lengths less than a given constant, although altogether there may be an infinite number of extremals joining P to Q among the extremals of all lengths. For a non-specialized pair of points the number of conjugate points of P prior to Q will be called the *type number* of g. We have the following theorem.

If on the extremals joining two non-specialized points P and Q, J is bounded, then there will be at most a finite set of extremals joining P to Q, and between the integers M_k which give the total number of extremals of type k joining P to Q, the following relations will hold true:

$$
\begin{aligned}
&1 \leqq M_0, \\
&1 \geqq M_0 - M_1, \\
&1 \leqq M_0 - M_1 + M_2, \\
(4)\quad &1 \geqq M_0 - M_1 + M_2 - M_3, \\
&\cdots\cdots\cdots\cdots\cdots, \\
&1 \leqq [M_0 - M_1 + M_2 - M_3 + \cdots + (-1)^{r-1} M_{r-1}](-1)^{r-1}, \\
&1 = M_0 - M_1 + M_2 - M_3 + \cdots + (-1)^r M_r.
\end{aligned}
$$

Simple examples will show that there may perfectly well be an infinite number of extremals joining two non-specialized points P and Q. Here the following more general theorem holds.

Corresponding to any positive constant a there exists a positive constant b with the following properties. If P and Q are two non-specialized points the relations (4) *will still be true if M_k be taken as the number of extremals of type k for which $J<a$, plus the number of extremals of type k in a suitably chosen subset of the extremals on which $a \leqq J < b$.*

Let P and Q be any pair of points, specialized or not. Let N_k be the number of extremals joining P to Q on which there are k points conjugate to P. If there are an infinity of such extremals we replace N_k by ∞. The sequence $N_0, N_1, N_2, \cdots$ will be called the *conjugate number sequence* for P and Q. If P and Q are non-specialized, we can readily prove the following.

If there are N_k extremals of type $k>1$, there are at least N_k extremals of the two adjacent types.

If there are N_1 extremals of type 1, there are at least N_1+1 extremals of the two adjacent types.

If there are N_0 extremals of type 0, there are at least N_0-1 extremals of the adjacent type 1.

These statements still hold if any of the N's are ∞, understanding $\infty-1$ or $\infty+1$ as standing merely for an infinity. Many other facts can be proved about the conjugate number sequence. One of the most curious as well as important facts is the following.

If P and Q are non-specialized and if any one of the symbols N_k is zero, then either at least two of the preceding symbols are ∞, or else the set of preceding symbols satisfies (4).

Suppose P and Q are any two points on S, specialized or non-specialized. The extremals joining P to Q may very well be infinite in number. In any case the extremals which joint P to Q, and for which J is less than a constant J_0, will either be finite in number, or else can be grouped into a finite set of r-parameter, connected, and in general analytic families, for which $r<m-1$, and on which J is a constant. Each isolated extremal joining P to Q to which a type number has not already been assigned, and each family of extremals joining P to Q, can be counted as if it were a finite set of extremals of definite types, and so counted all the preceding relations between the integers M_k still hold.

A strong aid in proving the existence of extremals joining any two points P and Q, whether specialized or not, is the following principle.

If in the respective neighborhoods of two points P_0 and Q_0, no matter how small the neighborhoods, there are two non-specialized points P and Q which are joined by at least one extremal of type k on which $J<J_0$, then there will be at least one extremal joining P_0 to Q_0 on which there are at least k points and at most $k+m-1$ points conjugate to P_0.

With the aid of this principle we have been able, in the examples which are to follow, to give strong existence

theorems which hold not only for non-specialized points, but also for specialized points as well.

7. *An Example.* Suppose we have an analytic surface S, with boundary b, and homeomorphic to a circular disc. On S consider the integral of arc length. Suppose S is extremal convex. Suppose S possesses a knob-shaped protuberance. More exactly suppose there is a portion of S homeomorphic to a disc, and bounded by a closed geodesic g that is shorter than nearby closed curves.

Any two points P and Q near g, or between g and b, will be connected by an infinite number of geodesics of type zero. By the aid of the preceding theorems one can then prove the following.

There will be an infinite number of geodesics joining P and Q upon each of which there is at least one point conjugate to P.

The later geodesics will include geodesics of arbitrarily great length which pass out of the region between g and b, and out of the neighborhood of g, thereby necessarily crossing the knob in some way.

This example suggests the more general question as to what effect the existence of a closed extremal in S will have upon the nature of extremals joining two points in S. If one admits closed extremals with conjugate points on them in varying numbers, and considers the problem in the different dimensions, one finds both curious and varied effects. The results are intimately connected with the so-called Poincaré rotation number.

8. *Extremals on General Regions.* The study of extremals in regions of general connectivities has been undertaken as yet only in a preliminary way. However, one case, namely the case of a region Σ homeomorphic with a portion of m-space between two concentric $(m-1)$-spheres has been studied, and has yielded much light. As in the case of the earlier region S so here we suppose Σ extremal-convex. A first result is the following.

If P and Q are any two non-specialized points on Σ, there exists

at least one extremal joining P to Q of every type k congruent to zero mod $m-2$.

It is a curious fact that, while this theorem affirms the existence of an infinite number of extremals when $m>2$, it affirms the existence of but one extremal when $m=2$. However, the gap is filled by the affirmation that when $m=2$ there is always an infinite number of extremals joining P to Q of the one type zero.

Concerning any pair of points P and Q, whether specialized or not, we can prove the following.

In the conjugate number sequence for any two points P and Q whatsoever there can nowhere be more than $2m-4$ consecutive zeros $(m>2)$.

In proving this theorem we have proved the a priori existence of an infinite set of extremals. This theorem shows that both the region and the dimension notably affect the results. It affords a contrast between the search for extremals that give minima and the search for extremals in general. Of the infinite set of extremals hereby affirmed to exist at most the first can give a minimum.

A simple example of an integral and extremal convex region such as Σ is the following. As the integral we take

$$\int \frac{1+(r-1)^2}{r}\,ds,$$

where ds is the differential of the arc length in m-space, and r the distance to the origin. The region between the two $(m-1)$-spheres, $r=a$ and $r=b$, is extremal-convex if we take $0<a<1<b$. It is easy to verify our theorems for any two points on the $(m-1)$-sphere, $r=1$.

9. *Extremals on Closed Manifolds.* Here the simplest and most important type of manifold has been investigated, namely an analytic manifold Σ homeomorphic to an m-sphere. Among others, the following two existence theorems have been proved.

If P and Q are any two non-specialized points on Σ, there

exists at least one extremal joining P to Q of every type k congruent to zero, mod $m-1$.

In the conjugate number sequence for any two points P and Q whatsoever there can nowhere be more than $2m-3$ *consecutive zeros* ($m>1$).

In proving the first of these theorems there is introduced a new method which might be called *the method of continuation of extremals by homeomorphisms*. First the existence of non-bounding r-families is gone into on the m-sphere. The greatest aids here are apparently the fundamental theorems on the characterization of conjugate points by deformations, and the knowledge of all the facts about geodesics on m-spheres. Once the existence of the non-bounding r-families is proved for the m-sphere it is of course proved for Σ. The existence of the required extremals then follows on Σ.

If Σ were any analytic closed manifold whatsoever, the method would require the selection of some manifold Σ_1 homeomorphic to Σ, upon which the study of extremals was as simple as possible. From facts about Σ_1 essentially belonging to the calculus of variations one would pass to facts about r-families, that is, facts essentially of analysis situs. From these last facts one would finally pass to the theorems about extremals on Σ.

This method may be contrasted with the much-used method of analytic continuation of extremals with respect to a parameter. Here one may be kept from the ultimate facts by the impossibility or difficulty of continuation beyond a certain point, or, in case one deals with an infinite set of extremals, with difficult questions of uniformity. All such difficulties as these are foreign to the above methods of continuation by homeomorphisms.

10. *The Case* $m=2$.* The case $m=2$ is essentially general in so far as deformations in the small are concerned so that the deformation theory given in the paper on the case $m=2$ needs practically no change to hold for the general case. A

* Morse II, loc. cit.

peculiarity of the case $m=2$ is that in an extremal convex region there can never be more than a finite number of extremals joining two points among the extremals with lengths less than a finite constant.

In the case $m=2$ it appeared desirable in a first paper, in the part that dealt with the theory in the large, to assume that the region was covered by a proper field of extremals, and to assume the extremals reversible. This had as a consequence that there were at most a finite set of extremals in S joining two points, but what was more important it reduced the essential passage from the small to the large to a matter of about three pages.* The problem of the extremals joining two fixed points when $m=2$ under this special field hypothesis has been taken up at the author's suggestion by D. E. Richmond† with a view to finding how complete were the relations already found. By methods peculiar to the special case treated he has established the completeness of the relations up to a certain point, and has found certain additional restrictive relations of a very elegant nature.

It is of interest to state in this connection that the author has proved his own relations between the numbers of critical points of a function and the connectivities of a general region are complete in general, and turning to the calculus of variations has proved his relations complete for the theory of extremals from a point to a manifold, and this under general hypotheses and for a general m. The method of proving a set of relations complete has been to take any set of integers M_k which satisfy the author's relations, and to set up an example to which these integers M_k belong.

11. *Closed Extremals.*‡ Closed extremals have been treated by the author up to the present time only for the case $m=2$.

* Morse II, loc, cit., pp. 227–231.

† Richmond, *Number relations between types of extremals joining a pair of points*, American Journal of Mathematics, vol. 50 (1928), pp. 370–388; and *A new proof of certain relations of Morse in the calculus of variations in the large*, which will appear in this Bulletin, Mch.-Apr., 1929.

‡ Morse II, loc. cit. Also Hadamard, loc. cit.

Here we start by cutting across the given closed extremal g with n perpendiculars whose intersections with g follow each other cyclically, and are so numerous and so near together that the resulting n segments of g have no pairs of conjugate points on them. We let $(u_1, \cdots, u_n) = (u)$ be distances from g measured respectively along these n perpendiculars, and we set up the closed broken extremal E joining successively the points on these perpendiculars at the respective distances (u) from g. Let $J(u)$ be the value of J along E.

As previously $J(u)$ will have a critical point when $(u) = (0)$. But here the quadratic form H of the terms of second order of $J(u)$ in an expansion about $(u) = (0)$ has marked differences from the form associated with an extremal joining two fixed points. For the purposes of the following theorems suppose g mapped conformally on a segment g_1 of the x axis of length ω, and this by a transformation which has the period ω in x.

The rank of H is n, $n-1$, or $n-2$, according as the Jacobi differential equation corresponding to g_1 has respectively no solutions of period ω not identically zero, has a one-parameter family of solutions of period ω not identically zero, or has no solutions other than those of period ω.

When one comes to the type number of H one finds another difference from the earlier results in that this type number does not depend solely on the distribution of conjugate points. In case the rank of H is n the type number depends upon a further classification of closed extremals as follows.

If $x = x_0$ is not conjugate to $x_0 + \omega$, then in the xy plane any point (x_0, a), $a > 0$, neighboring $(x_0, 0)$ can be joined to $(x_0 + \omega, a)$ by an extremal segment g'. The segment of g from x_0 to $x_0 + \omega$ will be said to be relatively *convex* or *concave* according as the slope of g' at $x_0 + \omega$ is greater or less than the slope of g' at x_0. In case the rank of H is n these slopes will not be equal, and in this case the type number of H is now determined as follows.

Let m be the number of points conjugate to x_0 preceding

$x_0+\omega$. When x_0 is not conjugate to $x_0+\omega$, the type number of H is m if g is convex, and $m+1$ if g is concave. When x_0 is conjugate to $x_0+\omega$, the type number is m.

In case the rank of H is $n-1$ or $n-2$, and the given closed extremal is isolated, it can be shown that for all subsequent purposes g may be counted as equivalent to a finite set of closed extremals for which the rank of H is n.

Closed extremals were considered in the large in a ring-shaped region bounded by two simple closed curves. For such a region the theory in the large is not essentially different from that previously given and will not be elaborated here.

There remains the important study of closed extremals in the higher spaces,* and on closed, as well as open, manifolds. Here even the most casual inspection discloses a wide and varied store.

12. *Extremals through a Point O cut Transversally† by a Closed Manifold Σ.* We here make the same hypotheses with respect to the integral as in §1. In the domain of definition of F we suppose we have a closed, analytic, regular,‡ $(m-1)$-dimensional manifold Σ. We suppose also that Σ is so placed in the field of extremals through O that there are no conjugate points to O on the extremals from O to Σ until after their intersections with Σ.

Let g be an extremal through O cut transversally by Σ at P. Let (v) be a set of $m-1$ parameters in an admissible‡

* For a first remarkable difference see Carathéodory, *Über geschlossene Extremalen und periodische Variationsprobleme in der Ebene und im Raume*, Annali di Matematica, (4), vol. 2 (1925), pp. 297–320. Here among other things it is shown that the Poincaré Law does not hold in the higher spaces.

† An interesting paper in this connection is that by White, M. B., *The dependence of focal points upon curvature for problems of the calculus of variations in space*, Transactions of this Society, vol. 13 (1912), pp. 175–198. This paper is primarily concerned with the differential and geometric nature of focal points in the small, in three-space, in particular in connection with the search for minima. No use is made of these results by the author.

‡ An analytic manifold will be called regular if on it the coordinates (x) of points neighboring any given point admit a representation in terms of $m-1$ parameters in which not all of the jacobians of $m-1$ of the co-

representation of Σ near P. On the extremals near g cut transversally by Σ at points (v), the coordinates (x) will be analytic functions of the parameters (v), and of the arc length s measured on these extremals from Σ. Let $\Delta(s)$ be the jacobian of these m coordinates (x) with respect to (v) and s, evaluated for the set $(v)=(v)_0$ that corresponds to P.

The points on g different from P at which $\Delta(s)=0$ are called the focal points of P on g, and the order of vanishing of $\Delta(s)$ at these points the order of the focal points.

The value of J taken from O to the points (v) will be indicated by $J(v)$. It appears first that $J(v)$ has a critical point when $(v)=(v)_0$. Let H be the quadratic form of the terms of second order in the expansion of $J(v)$ about $(v)=(v)_0$. Set $m-1=n$. We have the following theorem.

If O is a focal point of P of the rth order, the rank of the matrix of the form H is $n-r$. If O is not a focal point of P, the type number of H equals the number of focal points of P between P and O, always counting focal points according to their orders.

From the author's relations* between the critical points of a function we then obtain the following theorem in the large.

Suppose O is not a focal point of Σ, nor on Σ. Of the extremals from O to Σ which are cut transversally by Σ at their ends P on Σ, let M_i be the number upon which there are i focal points of P between O and P. Then between the numbers M_i and the connectivities R_i of Σ the following relations hold:

$$
\begin{aligned}
&M_0 \geqq 1 + (R_0 - 1),\\
&M_0 - M_1 \leqq 1 + (R_0 - 1) - (R_1 - 1),\\
&M_0 - M_1 + M_2 \geqq 1 + (R_0 - 1) - (R_1 - 1) + (R_2 - 1),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&M_0 - M_1 + \cdots + (-1)^n M_n\\
&\qquad = 1 + (R_0 - 1) - + \cdots + (-1)^n (R_n - 1).
\end{aligned}
\tag{5}
$$

ordinates with respect to the $m-1$ parameters are zero at once. Such a representation is called admissible.

* See Morse I, and Morse, *The analysis and analysis situs of regular n-spreads in $(n+r)$-space*, Proceedings of the National Academy, vol. 13 (1927), pp. 813–817.

If O is on Σ the above relations still hold if we count O as a nul-extremal of type zero.

The following corollary furnishes a strong existence theorem.

If O is not a focal point of Σ, the number of extremals from O cut transversally by Σ on which there are k focal points of Σ, is at least $R_k - 1$. The total number of extremals is at least

$$1 + (R_0 - 1) + \cdots + (R_n - 1). \tag{6}$$

The theorem and corollary apply at once to the problem of the number of normals from O to Σ, as follows.

If O is not a center of principal normal curvature of Σ, and if M_i is the number of normals from O to Σ on which there are i centers of principal normal curvature of Σ between O and the feet of the normals on Σ, then the relations (5) *hold.*

For example, if Σ is an admissible surface homeomorphic with a torus it follows from (6) there will be at least four normals to Σ from any point not a center of principal normal curvature. If Σ is a surface in 4-space homeomorphic with the complex obtained by identifying the opposite faces of an ordinary cube, one sees from (6) that there will be at least eight normals from a point not a center of principal normal curvature. It is possible to multiply these examples in any space. Moreover, even when O is a center of principal normal curvature, it is still possible to get strong existence theorems by a limiting process.

The question of the number of $(m-1)$-planes tangent to Σ through a given $(m-2)$-plane is closely related to the preceding theory on normals, and it is easy to proceed to existence theorems on the tangency of this $(m-1)$-plane. In fact there is a whole theory of the relations of straight lines, hyperplanes and hyperspheres to closed manifolds that can be treated after the spirit of this section.

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THE FORMS $ax^2+by^2+cz^2$ WHICH REPRESENT ALL INTEGERS

BY L. E. DICKSON

THEOREM. *$f=ax^2+by^2+cz^2$ represents all integers, positive, negative, or zero, if and only if*: I. *a, b, c are not all of like sign and no one is zero*; II. *no two of a, b, c have a common odd prime factor*; III. *either a, b, c are all odd, or two are odd and one is double an odd*; IV. *$-bc$, $-ac$, $-ab$ are quadratic residues of a, b, c, respectively.*

We shall first prove that I-IV are necessary conditions. Let therefore f represent all integers. It is well known that I follows readily.

If a and b are divisible by the odd prime p, f represents only $1+\frac{1}{2}(p-1)$ incongruent residues cz^2 modulo p. This proves II.

Next, *no one of a, b, c is divisible by* 8. Let $a\equiv 0 \pmod 8$. Every square is $\equiv 0$, 1, or 4 (mod 8). First, let $b=2B$. Since f represents odd integers, c is odd. Since $by^2\equiv 0$ or $2B$ (mod 8) and $cz^2\equiv 0$, c, or $4c$, f has at most six residues modulo 8. If m is a missing residue, f represents no $m+pn$. Second let b and c be odd. Then $4b\equiv 4c\equiv 4 \pmod 8$. Thus the residues of f modulo 8 are obtained by adding each of 0, 4, b to each of 0, 4, c; we get only seven residues 0, 4, b, c, $4+b$, $4+c$, $b+c$.

No one of a, b, c is divisible by 4. Let a be divisible by 4. Since a is not divisible by 8, $a\equiv 4 \pmod 8$. Evidently $f\equiv 0$, b, c, or $b+c \pmod 4$. No two of these are congruent modulo 4. If $b\equiv \pm 1 \pmod 4$, they are 0, ± 1, c, $c\pm 1$. Evidently c is not congruent to 0, ± 1, or ∓ 1. Hence $c\equiv 2 \pmod 4$. Since $b\not\equiv 0$, this proves that one of b and c is $\equiv 2 \pmod 4$. By symmetry, we may take $b\equiv 2 \pmod 4$. If $b\equiv 6 \pmod 8$, we apply our discussion to $-f$ instead of

f. Hence take $b \equiv 2 \pmod 8$. Thus $a \equiv 8n+4$, $b = 8m+2$, and c is odd. Since $x^2 \equiv 0$ or $1 \pmod 4$, $ax^2 \equiv 0$ or $8n+4 \pmod{16}$. Since $y^2 \equiv 0$, 1 or 4 (mod 8), $by^2 \equiv 0$, $8m+2$, or 8 (mod 16). We employ only even residues of f modulo 16. Then z is even, and $cz^2 \equiv 0$ or $4c \pmod{16}$. But $c \equiv \pm 1 \pmod 4$, $4c \equiv \pm 4 \pmod{16}$. Evidently ax^2+by^2 has at most 2×3 residues modulo 16. The missing two even residues are seen to be s and $s+4$, where $s = 10$ if n and m are both even, $s = 2$ if n is even and m odd, $s = 6$ if n is odd and m even, $s = 14$ if n and m are both odd. According as $4c \equiv 4$ or -4, f is not congruent to $s+4$ or s modulo 16.

No two of a, b, c are even. Let us set $a = 2A$, $b = 2B$. By the preceding result, A and B are odd. Also, c is odd. If $A = 4n-1$, we use $-f$ in place of f. Hence let $A = 4n+1$. Then $f \equiv 2x^2+2By^2+cz^2 \pmod 8$. Consider only odd residues of f. Then $cz^2 \equiv c \pmod 8$. The residues of $2x^2+2By^2$ are 0, 2, $2B$, $2B+2$. When these are increased by c, the sums must give the four odd residues modulo 8. Hence no two are congruent. Thus no two of 0, 1, B, $B+1$ are congruent modulo 4. Since B is odd and $\not\equiv 1 \pmod 4$, $B \equiv 3$, $B+1 \equiv 0$, a contradiction.

This completes the proof of property III. Properties II and III imply the following property.

V. *a, b, c are relatively prime in pairs.*

Thus $cd \equiv -b \pmod a$ has a solution d which is prime to a. Suppose that d were a quadratic non-residue of an odd prime factor p of a. Write $a = pA$. Consider values of x, y, z for which f is divisible by p. Then $z^2 \equiv dy^2 \pmod p$, whence y and z are divisible by p. Hence $f = pF$, where $F \equiv Ax^2 \pmod p$. Evidently Ax^2 takes at most $1+\frac{1}{2}(p-1)$ values incongruent modulo p. Hence there is an integer N that is not congruent to one of them. Thus f fails to represent $p(N+pw)$ for any value of w. This contradiction proves that $v^2 \equiv d \pmod p$ is solvable. The usual induction shows that it is solvable modulo p^n. Also, $d^2 \equiv d \pmod 2$. By means of the Chinese remainder theorem, we see that $w^2 \equiv d \pmod a$, is solvable whether a is odd or double an odd integer,

Then w is prime to a since d is. Since $(cw)^2 \equiv -bc \pmod{a}$ this proves IV.

We shall now prove that I–IV imply that f represents every integer g. It is known* that I, IV and V imply that $f=0$ has solutions x', y', z' which are relatively prime in pairs. Then the greatest common divisor of the three numbers $\alpha=ax'$, $\beta=by'$, $\gamma=cz'$ is 1. For, if they are all divisible by a prime p, one of x', y', z' is divisible by p (otherwise a, b, c would all be divisble by p). By symmetry, let x' be divisble by p. Then neither y' nor z' is divisible by p. Hence b and c would be divisble by p, contrary to V. Hence† if D is any given integer, ξ, η, ζ may be chosen so that

$$\alpha\xi + \beta\eta + \gamma\zeta = D. \tag{1}$$

We seek a solution of $f=g$ of the form

$$x = nx' + \xi, \qquad y = ny' + \eta, \qquad z = nz' + \zeta. \tag{2}$$

Since $ax'^2 + \cdots = 0$, $f=g$ is satisfied if

$$2Dn = g - e, \tag{3}$$

where

$$e = a\xi^2 + b\eta^2 + c\zeta^2. \tag{4}$$

If ξ', η', ζ' is a second set of solutions of (1), write

$$X = \xi - \xi', \qquad Y = \eta - \eta', \qquad Z = \zeta - \zeta'.$$

Then

$$\alpha X + \beta Y + \gamma Z = 0. \tag{5}$$

We seek the general solution of (5). Let δ be the greatest common divisor of $\alpha=\delta A$ and $\beta=\delta B$. Then δ is prime to γ, whence $Z=-\delta w$. Hence

$$AX + BY = \gamma w, \quad A, B \text{ relatively prime.} \tag{6}$$

There exist integers r, s satisfying

$$Ar + Bs = 1. \tag{7}$$

* Dirichlet-Dedekind, *Zahlentheorie*, ed. 4, §157, p. 432 (Supplement X).

† Since the g. c. d. 1 of α, β, γ, is a linear function of them. Multiply the relation by D.

Multiply the second member of (6) by (7). Thus

$$A(X - \gamma rw) + B(Y - \gamma sw) = 0.$$

The quantities in parenthesis are equal to Bm and $-Am$, where m is an integer. The resulting values of X and Y, together with $Z = -\delta w$, give the general solution of (6). Hence if ξ', η', ζ' is one solution of (1), the general solution is

$$(8)\quad \xi = \xi' + \gamma rw + Bm, \quad \eta = \eta' + \gamma sw - Am, \quad \zeta = \zeta' - \delta w,$$

where w and m are arbitrary, while r, s satisfy (7).

First, let a, b, c be all odd. Then $x'+y'+z' \equiv 0 \pmod 2$. But x', y', z' are not all even. Hence just one of them is even. By symmetry, we may take x' even, y' and z' odd. Then α is even, β and γ are odd, δ is odd, A is even, B is odd. Write

$$(9)\qquad e' = a\xi'^2 + b\eta'^2 + c\zeta'^2.$$

When working modulo 2, we may discard the exponents 2 in (4) and (9). Take $w=0$, $m=1$. Then, by (8),

$$\xi \equiv \xi' + 1, \quad \eta \equiv \eta', \quad \zeta \equiv \zeta', \qquad (\text{mod } 2),$$

$$e \equiv \xi + \eta + \zeta \equiv e' + 1.$$

For $w=m=0$, evidently $e=e'$. Hence we may take $e \equiv g \pmod 2$. We may take $D=1$. Then (3) yields an integral value of n. Hence $f=g$ is solvable.

Second, let a and b be odd, but c the double of an odd integer, whence $c \equiv 2 \pmod 4$. Since $f \equiv x+y \pmod 2$, $x'+y'$ is even. But x' and y' are relatively prime. Hence x' and y' are odd. Thus α and β are odd, γ is even, δ is odd, A and B are odd. By (1) and (4),

$$(10)\qquad D \equiv \xi + \eta \equiv e, \qquad (\text{mod } 2).$$

If g is odd, we take $D=1$. (See footnote on p. 59.) By (10), $g-e$ is even and (3) yields an integer n.

But if g is even, we take* $D=2$. By (10), $\xi+\eta$ and e are even. In (8), take $w=0$, $m=1$. Then

$$\xi = \xi' + B, \quad \eta = \eta' - A, \quad \zeta = \zeta'. \tag{11}$$

In case ξ' and η' are odd, we replace ξ', η', ζ' by the preceding solution having ξ and η even. Hence we may choose the initial solution ξ', η', ζ' so that ξ' and η' are even. Then (11) gives

$$e \equiv e' + aB^2 + bA^2 \equiv e' + a + b \pmod 4.$$

Hence if $a+b\equiv 2 \pmod 4$, we may choose e so that $e\equiv g \pmod 4$. Then (3) yields an integral value of n. But if $a+b\equiv 0 \pmod 4$, we take $w=1$, $m=0$ in (8) and see that ξ and η are even since γ is even. Then since δ is odd and $c\equiv 2 \pmod 4$,

$$e \equiv 2\zeta^2 = 2(\zeta' - \delta)^2 \equiv 2\zeta'^2 + 2 = e' + 2 \pmod 4.$$

As before, $f=g$ is solvable.

COROLLARY. *If $ax^2+by^2+cz^2$ is not a Null form, it does not represent all integers.*

Examples with $a=1$, $c=-C$, $C>0$.

(i) $b=1$. Then C must be odd or double an odd integer and -1 must be a quadratic residue of C. Then every odd prime factor of C is $\equiv 1 \pmod 4$. A necessary and sufficient condition on C is that it be a sum of two relatively prime squares.

(ii) $b=2$. Then C must be odd and -2 a quadratic residue of C. Then its prime factors are $\equiv 1$ or $3 \pmod 8$. A necessary and sufficient condition on C is that it be of the form r^2+2s^2, r odd, r and s relatively prime.

(iii) $b=3$. Then C must be odd or double an odd integer, C prime to 3, while C and -3 must be quadratic residues of each other. Hence every prime factor of C is $\equiv 1 \pmod 6$. Necessary and sufficient conditions on C are that C be odd and of the form r^2+3s^2, where r and s are relatively prime.

THE UNIVERSITY OF CHICAGO

* Elimination of ξ, η, ζ between (1) and (2) gives $\alpha x+\beta y+\gamma z=D$. Here $D\equiv x+y\equiv f \pmod 2$. Hence $D\equiv g \pmod 2$.

THE THEORY OF MULTIPLE IMPLICATION AND ITS APPLICATION TO THE GENERALIZED PROBLEM OF EPIMENIDES

BY H. B. SMITH

Let us allow $p, q, r, \cdots$ to stand for propositional functions, that is, for variables whose truth-values are in general dependent on the meaning of the terms that enter into them, and let the expression $(p\ q\ r \ldots)$ have the following verbal interpretation:

"$p, q, r, \cdots$ are simultaneously true for some meanings of the terms that enter into them."

We shall speak of this expression, $(p\ q\ r \cdots)$, or its negative, as the *existential* of the *n elements*, $p, q, r, \cdots$, and we shall say that it is of the first *order* and *n*th *degree*. The function containing existentials of the first order or their negatives as elements, will be of the second order and so on. For the particular case of one variable we should write:

(p) $= p$ is true for some meanings of the terms,

$(p)'$ $= p$ is true for no meanings of the terms,

$(p')' = p$ is true for all meanings of the terms,

(p') $= p$ is true for not all meanings of the terms,

the prime being used as the symbol of denial or negation.

The effect of the symbol ( ), regarded as an operator, will be in general to weaken the expression on which it operates, unless this expression be zero (o) or one (i). Thus,

p implies (p),

$(p')'$ implies p,

but not conversely. This simple provision is important, for it will lead at once to the result that certain generally recognized principles are untrue in a logic of complete generality. Thus it will turn out that the equality,

p implies $q = p'$ or q

does not hold true without limitation, namely, the limitation that the variables stand only for "propositions," for zero or one values, and not for propositional functions. This limitation is one that reduces the generality of our science and is directly related to an application of our theory which we shall make before we end. If we write,

$$p < q = p \quad \text{implies} \quad q,$$

we shall be prepared to set down the following definition,

$$(pq)' = p < q',$$

and from this follows the law of expansion for the general case,

$$(pqr\cdots)' = pq < r' + \cdots = p < q' + r' + \cdots,$$

that is, the elements are conceived to be conjoined as in a product, so that the consequent becomes the sum of the contradictories of the separate parts.

We might, if we liked, derive the fundamental laws of the calculus of propositions by assuming certain very simple properties of the existential, together with the fundamental equations,

$$(pq) = pq + p'q'(p)(q) + (p)(q')' + (q)(p')',$$
$$(pq)' = (p)' + (q)' + p'q(q') + q'p(p'),$$
$$(\overline{pq}) = (p') + (q'),$$
$$(\overline{pq})' = (p')'(q')',$$

the bar over pq standing for the negation of the product, or $p'+q'$. Or, we can derive the simple properties of the existential from the laws of the calculus of propositions. Obviously we may raise the degree of the function without changing its value, by adding to it as many one-elements as we choose. This fact is important, because it will enable us to provide that the symbol of implication can always be made to appear in our development. Thus $(o'=i)$,

$$(p)' = (ip)' = i < p',$$
$$(p)' = (pi)' = p < o.$$

And it is clear, too, that the elements may be permuted in any order, for

$$(p\,q)' = p < q' = q < p' = (qp)',$$

and that no change will result if we add to the existential as many elements already contained in it as we like. Similarly, the appearance of a pair of contradictory elements is a condition for the vanishing of the existential, if it be not negated; otherwise a condition for its becoming unity.

We approach now the consideration of a matter of the first importance, involving as it does, the creation of an infinite series of meanings of the "true" and the "false," together with an indication of how each one of these distinctions can be uniquely defined. We assume as true, for *all* meanings of the terms and for all meanings of p,

$$p < (p);$$

and we assert, moreover, that the converse,

$$(p) < p,$$

is true for *not all* meanings of the terms.

Now by allowing p to take on the values p', (p), $(p)'$, etc. and by the contradiction and interchange of antecedents and consequents, we should obtain a series of results that express the relation of the existentials of first and second order and beyond to one another and to the free variable within the propositional universe. These results are represented diagrammatically in the following table, the longer space being understood to "contain" the smaller spaces which it extends beyond.

| | | |

We are now in a position to define each one of these meanings of "true" and "false." If we write

$$f(p) = p \text{ is false},$$
$$t(p) = p \text{ is true},$$

it will be clear that $t=f'$, no matter what value f may have, and that, consequently, t will be unambiguous as soon as f has been defined.

Suppose, for example, we wish $f(p)$ to mean p'. It will be enough to require f to satisfy the equation

$$f(f) = f',$$

for while there is an infinite series of meanings of "false" that satisfy the implication

$$f(f) < f',$$

for example,

$$(p)', \quad ((p))', \quad ((p)'), \cdots,$$

and which do not satisfy the converse, and while there is an infinite series of distinct meanings of "false" that satisfy the implication

$$f' < f(f),$$

for example,

$$(p'), \quad ((p')), \quad ((p')')', \cdots,$$

and which do not satisfy the converse, there is only one meaning of "false," given by

$$f(p) = p',$$

which satisfies both implications at once. Accordingly $f=p'$ is unique and $t=p$.

Any meaning of "false" from among the members of the infinite series of meanings that this term can have, can be uniquely determined by requiring f to satisfy a given equation whose form can be found at once by inspection. Thus, if

$$f(p) = p < o, \qquad t(p) = (p < o)',$$

is the meaning of "true" and "false" desired, we assume f and t to satisfy respectively the equations

$$f(f) = f < o, \qquad t(t) = (t < o)',$$

for these solutions will prove to be unique. If

$$f(p) = (p' < o)', \qquad t(p) = p' < o,$$

are the meanings we choose, we require f and t to satisfy, respectively,

$$f(f) = (f' < o)', \qquad t(t) = t' < o,$$

and so on.

The unique solution of the equation $f(p) = o$ is $p = i$ for all meanings of f, as will appear as the result of a complete induction,

$$f(p) = p', \quad (p)', \quad (p'), \quad \text{etc.}$$

Hence p is true absolutely, that is, for all meanings of t,

$$t(p) = p, \quad (p), \quad (p')', \quad \text{etc.},$$

if f satisfy the implication

$$f < f(f).$$

The unique solution of the equation $f(p) = i$ is $p = o$ for all meanings of f. Hence p is false absolutely, if f satisfy the implication*

$$f(f) < f.$$

An interesting application of our theory will result if we should now go on to specialize the meaning of p and under-

* For further results that concern the theory of multiple implication see the writer's *Symbolic Logic*, (N. Y., Crofts, 1927). For a brief history of the problem of Epimenides as of other insolubilia see Professor E. R. Guthrie's admirable little monograph entitled *The Paradoxes of Mr. Russell*. This work, little known because privately printed, has not attracted the attention it deserves.

stand p to mean "this proposition." We should then have

$$f(p) = \text{this proposition is false;}$$

or, if we like,

$$1.f(1).$$

It is this situation that is supposed to give rise to a paradox, for it is supposed to lead to the identity of f with its own contradictory. Thus if 1. is o, $f(1)$ is i, if 1. is i, $f(1)$ is o and 1. and $f(1)$ are taken to be the same thing. The difficulty here depends on the assumption that every meaningful assertion must be specifically either zero or one, that is, must be a proposition, within Russell's understanding of that term. The solution to the difficulty will appear as soon as this limitation has been removed. Let us begin by defining the parts that enter into our assertion. The meaning of "false" has been defined already. It remains to define "this proposition."

We have here, perhaps, a range of choice in definition but let us require p to satisfy the equation

$$\{f < f(f)\} + \{f(f) < f\} = i.$$

If this condition be understood to stand for the meaning of p, it will be seen to have two unique solutions, $p = o$ and $p = i$. This, then, is the solution of the difficulty involved in the assertion

"this proposition is false,"

or, as we might indicate it,

$$1.f(1).$$

That is, if 1. is two-valued, being ambiguously zero or one, then $f(1)$ is similarly two-valued and no contradiction can arise.*

* We must not, of course, on this account assume the "identity" of p and $f(p)$. To do so would be to fall into the familiar fallacy of confusing the use of "all" in the collective and distributive sense. For all meanings (collectively) p and $f(p)$ are the same, but they are not equal for all meanings, where "all" has the meaning of "every."

The same solution can be shown to hold in the more general case. Consider a "cycle" of three assertions of the form

$$1.f(2) \qquad 2.f(3) \qquad 3.f(1)$$

We may now attach to these too a determinate and consistent meaning. From the symmetry of the situation we may take it for granted that each one has the same general character and that this general character is the one that we assumed above. That is, these three assertions are to be conceived as two-valued in the sense defined. Since 1, 2 and 3, then, are two-valued, so will the following be two-valued in the same sense, viz.

$$f(1) \qquad f(2) \qquad f(3)$$

and for two reasons. In the first place, because if p is two-valued so is $f(p)$, (for the condition which p is required to satisfy, may be regarded equally as the condition which $f(p)$ is required to satisfy, for all meanings of f, and there are two and only two such values, viz., $f(p)=o$ and $f(p)=i$), and in the second place, because they are repetitions of 1, 2 and 3. Consequently, no contradiction can arise from the simultaneous assertion of 1, 2 and 3. The same remarks will evidently hold of a "cycle" of any number of assertions of the same form and the generality of the method is evident.

THE UNIVERSITY OF PENNSYLVANIA

CONCERNING COVARIANTS OF THE RATIONAL PLANE QUARTIC CURVE WITH COMPOUND SINGULARITIES*

BY J. H. NEELLEY

1. *Introduction.* After investigating the invariants of the rational plane quartic curve R_2^4 with compound singularities† it is of interest to see what effect such singularities have on covariants of the curve. This paper observes the effect of the tacnode and the ramphoid cusp upon associated curves and sets of covariant parameters.

2. *Covariant Forms of the Curve with a Tacnode.* To examine these forms we let $t=\pm a$ be the tacnodal parameters and t_1 and t_2 the contacts of the distinct double line of the curve with an acnode. This may be done if 0 and ∞ are the contacts of the two tangents to the curve from the tacnode.‡ These tangents and the line through their contacts give the very neat representation

$$(1)\qquad \begin{cases} x_0 = t^4 - a^2t^2, \\ x_1 = t^2 - a^2, \\ x_2 = t^3 + st^2 + \sigma_2 t, \end{cases}$$

where $\sigma_1=t_1+t_2$, $\sigma_2=t_1t_2$, and $s=[a^2\sigma_1^2+(a^2+\sigma_2)^2]/(2a^2\sigma_1)$.

The pencil of conics§

$$(2)\qquad g_2 - \lambda K = 0,$$

where g_2 is the envelope of lines which cut the curve in self-apolar sets of points and K is the locus of the vertices

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* Presented to the Society, April 6, 1928.

† Neelley, *Compound singularities of the rational plane quartic curve*, American Journal of Mathematics, vol. 49 (1927), pp. 389–400.

‡ Neelley, loc. cit., p. 394.

§ J. E. Rowe, *Covariants and invariants of the rational plane quartic*, Transactions of this Society, vol. 12 (1911), pp. 298–299.

of the flex-triangles of the first osculants of R_2^4, is not affected by the singularity. It has one member which passes through the tacnode and that is the conic on the contacts of the double lines. This conic has the tacnodal line

$$x_0 - a^2 x_1 = 0 \tag{3}$$

as its tangent at the tacnode which is to be expected as several double line contacts coincide at a tacnode.* The tacnode does not affect the covariant line on the points q_i† nor the line which is the locus of a point such that tangents from it to R_2^4 are apolar to the flexes.†

Next we examine the combinants of two line sections (ξx) and (ηx) of the curve (1).‡ These combinants also seem to be independent of the tacnode except C which has $x_0 - a^2 x_1$ as a squared factor. This was expected as C gives the double lines of R_2^4. The fundamental involution§ determined by two binary quartics does not give any special information as none of the forms which make up the complete system‖ of these two quartics vanish identically.

3. *Covariant Forms of the Curve with a Ramphoid Cusp.* This singularity is a special form of the tacnode but it affects the covariants very strikingly. Suppose $t=0$ at the cusp and $t=\infty$ at a distinct real flex. There is always such a flex as three flexes coincide at the cusp.¶ Then the cuspidal line, the flex line and the join of cusp and flex give

$$\begin{cases} x_0 = at^4, \\ x_1 = 4bt^3 + 6ct^2, \\ x_2 = 4dt + e. \end{cases} \tag{4}$$

Two lines (ξx) and (ηx) cut out the line sections

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* Neelley, loc. cit., p. 395.
† Rowe, Transactions of this Society, vol. 13, p. 390.
‡ Salmon, *Modern Higher Algebra*, 4th ed., *p.* 296.
§ Meyer, *Apolarität und rationale Kurven*, p. 9.
‖ Salmon, loc. cit., p. 224.
¶ Neelley, loc. cit., p. 390.

$$(5)\qquad \begin{cases} U \equiv a\xi_0 t^4 + 4b\xi_1 t^3 + 6c\xi_1 t^2 + 4d\xi_2 t + e\xi_2 = 0, \\ V \equiv a\eta_0 t^4 + 4b\eta_1 t^3 + 6c\eta_1 t^2 + 4d\eta_2 t + e\eta_2 = 0. \end{cases}$$

The pencil of conics considered in §2 is derived from combinants of $U+\mu V$ by means of the rate of exchange

$$\begin{vmatrix} (a\xi) & (b\xi) \\ (a\eta) & (b\eta) \end{vmatrix} = \begin{vmatrix} a & & \\ & b & \\ & & x \end{vmatrix}^*,$$

and so on. Here g_2 has the equation

$$(6)\qquad 16b^2d^2x_0^2 + a^2e^2x_1^2 + 12ac^2ex_0x_2 + 8abdex_0x_1 = 0,$$

and K the equation

$$(7)\qquad b^2d^2x_0^2 + acd^2x_0x_1 + ac^2ex_0x_2 = 0.$$

Obviously g_2 is on the cusp (0, 0, 1) and has the line $x_0=0$ as its tangent at that point. Also K is degenerate with $x_0=0$ as one of its lines. So g_2 and K have contact at the cusp and the pencil is composed of conics having contact with R_2^4 at the cusp, all of the conics cutting out at least four coincident points of the quartic at that point. So the octavics cut out by the pencil of conics have four-fold roots. Then if $\lambda=12$ we have another degenerate member of the pencil. This is the Stahl conic N† which is the envelope of the flex lines of the cubic osculants of R_2^4. Also since the conic on the flexes and that on the contacts of the double lines are members of this pencil, we have means of proving the following known theorem.‡

THEOREM 1. *Three flexes and one of the points q_i coincide at a ramphoid cusp and two double lines coincide there.*

The pencil of covariant octavics cut out by the pencil is

$$(8)\qquad \begin{aligned} &(16-\lambda)b^2d^2t^8 + 4bd(8be - cd\lambda)t^7 \\ &\quad + 2(8b^2e^2 + 24bcde - 3c^2d^2\lambda)t^6 \\ &\quad + 4ce(12be + 3cd - cd\lambda)t^5 + c^2e^2(48-\lambda)t^4 = 0, \end{aligned}$$

* Rowe, loc. cit., p. 296.

† Stahl, *Ueber die rationale ebene Curve vierter Ordnung*, Journal für Mathematik, vol. 101 (1886), pp. 300–325.

‡ Neelley, loc. cit., p. 390 and p. 395; Wieleitner, *Theorie der ebenen algebraischen Kurven höherer Ordnung*, 1905, pp. 280–281.

which has several covariant forms that vanish identically due to the four-fold root.* When $\lambda = 48$, we have the conic on the contacts of the double lines and the octavic it cuts out has a five-fold root, and so a $C_{2,10}$† vanishes also.

The line on the points q_i is

$$8(bde + 3cd^2)x_0 - ae^2x_1 = 0, \tag{9}$$

and the locus of a point such that tangents from it to the curve are apolar to the flexes is the line

$$2(bde - 6cd^2)x_0 - ae^2x_1 = 0. \tag{10}$$

So these lines cut out a pencil of covariant quartics

$$[bde(1 - 5\lambda) - 6cd^2(1 + 3\lambda)]t^4 - 2be^2t^3 - 3ce^2t^2 = 0, \tag{11}$$

with at least a double root, and again we have covariant forms which vanish. One member is the square of a quadratic and so it has a $C_{3,6}$‡ which vanishes. This quartic is cut out by the distinct double line on the cusp. The cuspidal line gives a member of the pencil with a four-fold root and so several covariant forms vanish.‡ The bicombinant conic on the nodes and the points q_i also has contact with R_2^4 at the cusp and so cuts out another octavic with a four-fold root.

The Salmon combinant C§ of U and V gives the double lines of R_2^4 with the equation

$$\begin{aligned} &8b^4d^4x_0^4 + 4(6ab^2cd^4 - ab^3d^3e)x_0^3x_1 \\ &\quad + 2(ab^4e^3 + 9ab^2c^2d^2e - 6ab^3cde^2)x_0^3x_2 \\ &\quad + 6(3a^2c^2d^4 - a^2bcd^3e)x_0^2x_1^2 \\ &\quad + 3(a^2b^2ce^3 - 6a^2bc^2de^2 + 9a^2c^3d^2e)x_0^3x_1x_2 = 0, \end{aligned} \tag{12}$$

which gives the following theorem.

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* Tracey and Moore, *Covariant conditions for multiple roots of binary forms*, soon to appear in the American Journal of Mathematics.

† Cayley, Collected Mathematical Papers, vol. II, p. 316.

‡ Cayley, loc. cit., p. 555.

§ Salmon, loc. cit., p. 296.

THEOREM 2. *Two double lines coincide at a ramphoid cusp and a third double line is on the cusp, or five contacts of double lines coincide at a ramphoid cusp.*

The combinant D gives the flex lines, and their equation is of the form

$$acx_0^3 x_2 \cdot Q = 0, \tag{13}$$

where Q is a quadratic in x_0, x_1, x_2. This form proves the following theorem.

THEOREM 3. *Three flex lines unite to form the cuspidal line at a ramphoid cusp.*

We also find x_0 to be a factor of the combinant E which is the envelope of lines which cut R_2^4 in catalectic quartics.

This close connection between the cusp and covariant forms is very nicely brought out by considering the invariants of a pencil of line sections on the cusp. Suppose $(\xi x)=0$ is the cusp line at $t=0$ and $(\eta x)=0$ is on the cusp and also has contact with R_2^4 at $t=\infty$. Then

$$(a\xi)(\alpha t)^4 \equiv at^4, \quad (b\eta)(\beta t)^4 \equiv 6ct^2,$$

and so*

$$I_2 = I_2' = I_4 = I_6 = 0.$$

But I_2 is the conic K and I_2' is the conic N so these meet at the cusp. Also I_4 is R_2^4 itself and I_6 gives the flex lines. So a cusp can only be at an intersection of K and N with R_2^4 where a flex line cuts the curve.

The significant thing we note concerning these covariant forms is that they cut out covariant sets of parameter values which are given by binary forms with multiple roots. This multiplicity is such as to be characteristic of the curve with a ramphoid cusp.

Consideration of the group of covariant forms which constitute the complete system of the two binary quartics giving the fundamental involution of the curve shows that one

* Morley invariants of two binary quartics.

member of the system vanishes identically. This form is the quadratic form obtained by operating with the quartic

$$(\beta t)^4 \equiv 12bt^2 + 12ct$$

on the sextic $12(2, H_2)$, where 2 is the second quartic $(\gamma t)^4 \equiv 4dt^4 + 4et^3$ and H_2 is its Hessian. It seems that *the identical vanishing of this covariant form is a necessary and sufficient condition that R_2^4 should have a ramphoid cusp.* We arrive at this conclusion because this covariant does not vanish for any other singularity and its vanishing is independent of the reference scheme giving the curve.

This investigation gives one other theorem that is not known. We noted that the conic K is degenerate when the curve has a ramphoid cusp. Previously* it was pointed out that this was also true when the curve had one or more undulations. So it seems that K is degenerate if there is a line of R_2^4 which cuts out four points of the curve with the same parameter value. Besides the undulations and the ramphoid cusp, the only type of curve where this is true is the limiting form of the oscnode when one loop is drawn up so as to become a point. Any line on this point cuts out three points and the oscnodal line cuts out four. The conic K for this form is the square of the oscnodal line. So we do have the following thorem.

THEOREM 4. *The Stahl conic K degenerates if the quartic curve has a line which cuts out four coincident points with the same parameter value and this line is one of the lines of the conic. The types of quartics with two such lines, the curve with two undulations or an undulation and a ramphoid cusp, have K given by the product of the two lines. When there are three such lines the equation of K vanishes identically.*

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* Neelley, *A note on the rational plane quartic curve with cusps or undulations*, this Bulletin, vol. 34 (1928), pp. 639–645.

ON THE REMOVAL OF FOUR TERMS FROM AN EQUATION BY MEANS OF A TSCHIRNHAUS TRANSFORMATION*

BY RAYMOND GARVER

The removal of the second, third, and fourth terms from an equation $f(x)=0$ by means of a Tschirnhaus transformation is a matter which can be summarized in two statements. First, if $f(x)=0$ is of higher than the fourth degree, the well known Jerrard process, which uses a fourth degree transformation, serves to remove the three terms without requiring the solution of any equation of degree greater than three. Second, if $f(x)=0$ is of the fourth degree, Lagrange first showed that a similar reduction is possible, employing a third-degree transformation.† Lagrange's transformation does not, however, apply to higher degree equations; the fact that the transformed quartic is a binomial equation is essential to its success.

The removal of the second, third, fourth, and fifth terms leads to a larger number of different cases. In general, we can say that, since the equations of condition are of degree 1, 2, 3, 4, we may be led to an equation of the 24th degree. But this maximum can almost always be substantially reduced. Thus Hamilton, in his report on the validity of Jerrard's work, was able to show that, if $f(x)=0$ is of degree greater than ten, the transformation can be effected with the aid of a single quartic equation, and no equation of higher degree, while simply one additional quartic has to be introduced if $f(x)=0$ is of degree ten.‡ Sylvester later obtained

* Presented to the Society, December 27, 1928.

† See his Oeuvres, vol. III, Paris, 1869, pp. 284–295. The article, his well known *Réflexions sur la résolution algébrique des équations*, was first published in 1770–1771.

‡ Sixth Report of the British Association for the Advancement of Science, London, 1837, pp. 295–348. See, in particular, p. 318 ff.

these same results in a somewhat different manner.* Bohlin professed to show that the transformation for $f(x)=0$ of degree ten required only one quartic equation,† but Netto‡ and Heymann§ pointed out that his work was incorrect. Heymann also set up a very simple transformation for the removal of the first four intermediate terms which requires a single quartic, and no equation of higher degree, but it can be applied only if $f(x)=0$ is of at least the twelfth degree. The only other work on the subject seems to be that of Lagrange in connection with the quintic;‖ he showed that the four intermediate terms can be removed without requiring the solution of any equation of higher than the sixth degree. But his method applies only to the quintic, since it makes use of the fact that the transformed equation is binomial.

The purpose of the present paper is to consider the cases which do not seem to be covered in the literature, namely, those in which the equation $f(x)=0$ is of degree 6, 7, 8 or 9. The following theorem gives the principal result:

THEOREM. *The second, third, fourth, and fifth terms of a ninth degree equation can be removed by means of a Tschirnhaus transformation of the eighth degree, the determination of whose coefficients involves the solution of no equation of degree greater than five.*

The method used to obtain this result was suggested by an article of Hilbert.¶ However, the theorem does not at all follow from his work, which leads to an equation of the twenty-seventh degree. In fact the aim of Hilbert's paper is simply to show that the general ninth degree equation is solvable in terms of algebraic functions of not more than four arguments, in the same sense that the general quintic is solvable

* Journal für Mathematik, vol. 100 (1887), pp. 465–486.

† Arkiv för Matematik, Astronomi, och Fysik, vol. 3 (1906–7), No. 5.

‡ Grunert's Archiv, (3), vol. 14 (1909), pp. 125–126.

§ Grunert's Archiv, (3) vol. 19 (1912), pp. 320–330.

‖ Pp. 314–317 of the above reference.

¶ Mathematische Annalen, vol. 97 (1926), pp. 243–250.

in terms of functions of a single argument. The point of view is thus quite different from that of the present paper.

To remove the second, third, fourth, and fifth terms from an equation $f(x)=0$ of the ninth degree by a Tschirnhaus transformation $y=P(x)$, where $P(x)$ is a polynomial of not higher than the eighth degree, we must show that the coefficients of $P(x)$ can be chosen so as to make $\sum y=\sum y^2 =\sum y^3=\sum y^4=0$, where $\sum y$ means $\sum_{i=1}^{9} P(x_i)$. If we assume $P(x)$ to be a simple eighth degree polynomial, and attempt to set up the equations of condition directly, the work becomes involved. Hence we proceed as follows.

A theorem of Brioschi* states that, if $f(x)=0$ is of the nth degree, n odd, there exist $(n-1)/2$ polynomials in x, $\phi_1, \phi_2, \cdots, \phi_{(n-1)/2}$, such that (1) each ϕ_i is of lower degree than $f(x)$, (2) the ϕ_i are linearly independent, (3) the ϕ_i possess the properties $\sum\phi_i=0$†, $\sum\phi_i^2=0$, $\sum\phi_i\phi_j=0$, $(i, j=1, 2, \cdots, (n-1)/2, i\neq j)$, (4) the determination of the coefficients of the ϕ_i requires the solution of linear and quadratic equations only. Thus for $n=9$ there are 4 such functions, and if we set up the transformation

$$y = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + c_4\phi_4 \tag{1}$$

we have at once $\sum y=\sum y^2=0$, independent of the c_i. The two remaining equations of condition, $\sum y^3=0$ and $\sum y^4=0$, are clearly homogeneous equations of degree 3 and 4 respectively in the c_i, with coefficients which can be computed in terms of the coefficients of $f(x)=0$. If these two equations involved only two essential parameters their solution would, in general, require the solution of a 12th degree equation, but the presence of an additional parameter allows it to be carried out without introducing any equation of degree higher than the fifth.

A well known theorem of Sylvester says that the general homogeneous cubic polynomial in four variables may, with

* Rendiconti del Istituto Lombardo, (2), vol. 20 (1887), pp. 364–370. Also his Opere, vol. 3, Milan, 1904, pp. 293–299.

† $\sum\phi_i$ means $\sum_{k=1}^{n}\phi_i(x_k)$, where the x_k are the n roots of $f(x)=0$.

the aid of an equation of the fifth degree, be reduced to the canonical form $M_1^3+M_2^3+M_3^3+M_4^3+M_5^3$, where the M_i^3 are the roots of a quintic whose coefficients are rational in those of the given cubic polynomial, the M_i are linear homogeneous functions of the four variables, and M_5 can be expressed as a linear function of the other M_i, say as $\sum_{i=1}^{4} v_i M_i$. The condition $\sum y^3 = 0$ can then be put in the form

$$(2) \quad M_1^3 + M_2^3 + M_3^3 + M_4^3 + (v_1M_1 + v_2M_2 + v_3M_3 + v_4M_4)^3 = 0,$$

where each M is a linear homogeneous function of c_1, c_2, c_3 and c_4. We wish to show that this equation can be satisfied in such a way as to leave one essential parameter undetermined.

In the exceptional case in which v_1, v_2, v_3, v_4 are all zero, (2) can be satisfied by setting $M_2 = -M_1$, $M_4 = -M_3$. Thus only two linear conditions are imposed on the c_i, and one essential parameter (that is, the ratio of two c's) remains unfixed.

In the general case, at least one v, say v_3, will be different from zero. We may also certainly rearrange subscripts so that the conditions $v_1 = 0$, $v_2^3 = v_3^3$ do not hold simultaneously. Then make the following substitutions

$$(3) \quad \begin{cases} M_1 = s + 1, \\ M_2 = a_2 s + b_2, \\ M_3 = (-v_1 - v_2 a_2)s/v_3 + b_3, \\ M_4 = b_4. \end{cases}$$

Equation (2) then becomes a non-homogeneous cubic in s, which can be satisfied identically in s by proper choice of a_2, b_2, b_3, b_4. First, the coefficient of s^3 is

$$1 + a_2^3 - \left(\frac{v_1 + v_2 a_2}{v_3}\right)^3,$$

and a_2 can be chosen so that this vanishes, under the restrictions on v_1, v_2, v_3, stated at the beginning of this paragraph.

The coefficients of s^2 and s are found to be respectively three times the quantities

$$1 + a_2^2 b_2 + a_3^2 b_3,$$

$$1 + a_2 b_2^2 + a_3 b_3^2,$$

where a_3 stands for $-(v_1+v_2a_2)/v_3$. Now a_2 can be chosen different from zero unless $v_1^3 = v_3^3$, $v_2 = 0$. In this case, $a_3 = -v_1/v_3$, and the coefficients of s^2 and s can both be made to vanish by taking $b_3 = -v_1/v_3$. Secondly, a_3 will be zero if $a_2 = -v_1/v_2$; further, $1+a_2^3$ must equal zero, or $v_1^3 = v_2^3$. In this case both the above coefficients will vanish if $b_2 = -v_1/v_2$. Otherwise a_3 cannot equal zero unless $v_1 = v_2 = 0$; then for $a_2 = b_2 = -1$ the coefficients of s^3, s^2, s will all vanish. In the general case, both a_2 and a_3 will be different from zero, and b_2 and b_3 can obviously be determined so that the coefficients of s^2 and s will vanish.

Finally, the term free of s becomes

$$1 + b_2^3 + b_3^3 + b_4^3 + (v_1 + v_2 b_2 + v_3 b_3 + v_4 b_4)^3.$$

To make this vanish requires simply the solution of a cubic in b_4, unless $1+v_4^3 = 0$. If this condition holds, there remains a quadratic to solve unless $v_1+v_2b_2+v_3b_3 = 0$. However, if this last equation is satisfied, it may be combined with conditions already set up to show that $1+b_2^3+b_3^3$ must also equal zero. Thus the term free of s still vanishes, and the coefficients of (3) have now been determined so that (2) vanishes identically in s.

Now since the M_i are linear homogeneous functions of $c_1, \cdots, c_4$, the c_i may be determined linearly in terms of $M_1, \cdots, M_4$. In general, this determination will be unique, but this will not be so if there is a linear relation between the M_i. By (3), the c_i are given as linear, non-homogeneous functions of s. If now we substitute these values into the expression for $\sum y^4$, the final equation of condition, $\sum y^4 = 0$, becomes a non-homogeneous quartic equation in s, which can

be solved for s.* This completes the proof of the theorem.

If $f(x)=0$ is of degree 7 or 8,† there exist three ϕ-functions having the properties of Brioschi's theorem. We then set up a transformation similar to (1), and the equations $\sum y^3=0$, $\sum y^4=0$ are homogeneous equations in three parameters. The determination of c_1, c_2, c_3 will, in general, lead to an equation of the 12th degree, as was pointed out above.

If $f(x)=0$ is of degree 6 it does not seem possible to lower materially the maximum 24 which was obtained in the second paragraph.

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SOME PROPERTIES OF MULTI-COHERENT CONTINUA‡

BY W. A. WILSON

1. In a recent paper§ C. Kuratowski gives the following definitions:

A continuum C is called *uni-coherent* (or *n-coherent*) if for every decomposition of C into two continua K and L, where $C\neq K\neq L$, $K\cdot L$ has one (or n) components. A continuum C is called *multi-coherent*, if $K\cdot L$ is not connected.

The properties of such continua are later developed in some detail. Among the theorems proved are the following.

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* The very exceptional case in which the coefficients of s^4, s^3, s^2, s all vanish, while the term free of s does not vanish, may be handled by introducing homogeneous parameters in (3), that is, by putting $M=s+t$, $M_2=a_2s+b_2t$, etc. For the present case, t must be zero.

† Although Brioschi's theorem does not mention equations of even degree, $(n-1)/2$ ϕ-functions can be set up for an equation of even degree $n+1$ just as they can for an equation of odd degree n, and in different ways as well.

‡ Presented to the Society, October 27, 1928.

§ C. Kuratowski, *Sur la structure des frontières communes à deux régions*, Fundamenta Mathematicae, vol. 12, pp. 20–42.

(I) *For a continuum C to be multi-coherent it is necessary and sufficient that, for every sub-continuum Q of C, $C-Q$ is connected.*

(II) *If C is a bounded plane set which is the common frontier of two regions, C is a multi-coherent continuum.*

(III) *A bounded plane continuum which is decomposable and bi-coherent is the common frontier of two regions.*

A careless reading of the above definitions may give the erroneous impression that a multi-coherent continuum is n-coherent for some value of n. The continuum given in Ex. 4(a) (loc. cit., p. 23) is multi-coherent, but not n-coherent, since $K \cdot L$ may have either two or three components according to the manner of decomposition of C. Furthermore, as Kuratowski points out, the continuum just cited shows that the converse of Theorem III is not true. The converse of Theorem II also is not valid, and hence the class of multi-coherent plane continua is larger than that of regular frontiers.* Although the problem of finding necessary and sufficient intrinsic conditions for a bounded decomposable continuum to be a regular frontier has been solved,† less general conditions may at times be more convenient to use and the close relationship between multi-coherent continua and regular frontiers seems to the author to make it of interest to determine additional conditions under which the former class of continua have the same intrinsic properties as the latter.

Certain properties follow at once from the equivalent definition of multi-coherent continua given in Theorem I above. We first note that, if the bounded multi-coherent continuum M is the union of two proper sub-continua H and K, both H and K are irreducible about the set $H \cdot K$.

* A set is called a *regular frontier* if it is the frontier of at least two components of its complement.

† See C. Kuratowski, *Sur la séparation d'ensembles situés sur le plan*, Fundamenta Mathematicae, vol. 12, p. 235; and W. A. Wilson, *On bounded regular frontiers in the plane*, this Bulletin, vol. 34 (1928), p. 86.

For otherwise either H or K would contain a sub-continuum disconnecting M. (A continuum M is said to be disconnected by the set C if $M-C$ is non-void and is not connected.) Furthermore, if M is a bounded decomposable multi-coherent continuum, there is always an irreducible decomposition of M into two proper sub-continua H and K.* In this case no proper sub-set of $H \cdot K$ disconnects M and we may say that $H \cdot K$ irreducibly disconnects M.

2. THEOREM. *Let M be a bounded multi-coherent continuum. Let A and B be proper sub-continua of M which are not continua of condensation and let $A \cdot B = 0$. Then $A+B$ disconnects M.*

PROOF. Set $K = M-(A+B)$. If K is connected, $\overline{K}$ is a continuum. Now $A' = A - A \cdot \overline{K}$ and $B' = B - B \cdot \overline{K}$ are not void and $A \cdot B = 0$, by hypothesis. Hence $A'+B'$ is not connected. But $A'+B' = M-\overline{K}$. Hence we have the contradiction that the continuum $\overline{K}$ disconnects M. Therefore K is not connected.

3. THEOREM. *Let M be a bounded multi-coherent continuum. Let A and B be proper sub-continua of M, let $A \cdot B = 0$, and let $A+B$ disconnect M. Then $K = M-(A+B)$ has precisely two components.*

PROOF. Let C be a sub-continuum of M irreducible between A and B.† Then $C - C \cdot (A+B)$ is connected and has limiting points on both A and B. Hence at least one component k of K has limiting points on both A and B and $k+A+B$ is a continuum.

Since $A+B$ disconnects M, $K-k = M-(k+A+B)$ is not void. Since M is disconnected by no sub-continuum, $K-k$ is connected. The theorem is proved.

4. THEOREM. *Let M be a bounded multi-coherent continuum. Let A and B be proper sub-continua of M which are not continua*

* This is shown in the reference at the beginning of this section, p. 25, Theorem (η).

† This does not imply that C contains either A or B, but merely that $C \cdot A \neq 0 \neq C \cdot B$ and that there is no proper sub-continuum of C having this property.

of condensation and let $A \cdot B = 0$. *Let* $K = M - (A + B)$. *Then* $\overline{K}$ *is the sum of two continua which are irreducible between* A *and* B *and have no common points.*

PROOF. By §§2 and 3, K has two components, K' and K'', one of which, say K', has limiting points on both A and B. If K'' had limiting points on A only, we would have the contradiction that A disconnects M. Hence $\overline{K}'$ and $\overline{K}''$ are continua joining A and B and $\overline{K}' \cdot \overline{K}'' = 0$, since otherwise $\overline{K}' + \overline{K}''$ would disconnect M.

Let C and D be sub-continua of $\overline{K}'$ and $\overline{K}''$, respectively, irreducible between A and B. Then $C \supset K'$, for otherwise $C + A + B$ would disconnect M. Thus $C = \overline{K}'$ and in like manner $D = \overline{K}''$, which proves the theorem.

COROLLARY 1. *Let* M *be a bounded multi-coherent continuum. Let* M *contain two proper sub-continua which are not continua of condensation and have no common points. Then* M *is the union of four continua* A, B, C, *and* D, *such that* $A \cdot B = 0 = C \cdot D$, A *and* B *are irreducible between* C *and* D, *and* C *and* D *are irreducible between* A *and* B.

PROOF. Let A' and B' be proper sub-continua of M which are not continua of condensation and have no common points, and let $K = M - (A' + B')$. Then $\overline{K}$ is the sum of two continua C and D which are irreducible between A' and B' and have no common points. Let $L = M - (C + D)$. Then $\overline{L}$ is the sum of two continua A and B which are irreducible between C and D and have no common points. It is evident that $A \subset A'$, $B \subset B'$, and C and D are irreducible between A and B.

COROLLARY 2. *If* M *satisfies the hypotheses of Corollary* 1, *it is the union of two continua* H *and* K *such that* $H \cdot K$ *is the sum of two closed sets between which both* H *and* K *are irreducible.*

To see this set $H = A + C$ and $K = B + D$ in the above corollary.

COROLLARY 3. *If* M *satisfies the hypotheses of Corollary* 1

and is a plane continuum, it cuts the plane and is the frontier of exactly two components of its complement.

This conclusion follows from Corollary 2 and a theorem proved elsewhere.*

5. THEOREM. *Let M be a bounded multi-coherent continuum. Let A and B be proper sub-continua of M, let $A \cdot B = 0$, and let $A+B$ disconnect M. Let $L = M-(A+B)$. Then $\overline{L}$ is the union of two continua, which are irreducible between A and B and whose only common points lie on $A+B$.*

PROOF. Let h and k be the two components of L. Obviously $\overline{L} = \bar{h}+\bar{k}$. Set $H = \bar{h}$ and $K = \bar{k}$. As in the proof of §4 it can be shown that H and K are irreducible between A and B. Since $h = H - H \cdot (A+B)$ and $k = K - K \cdot (A+B)$, and $h \cdot k = 0$, it follows that $H \cdot K \subset A+B$.

COROLLARY 1. *Let M be a bounded multi-coherent continuum. Let A and B be continua of condensation of M, let $A \cdot B = 0$, and let $A+B$ disconnect M. Then M is the union of two continua H and K, $H \cdot K = \alpha+\beta$, where α and β are closed subsets of A and B respectively, and both H and K are irreducible between α and β.*

COROLLARY 2. *In addition to the hypotheses of Corollary* 1 *let M be a plane continuum. Then M cuts the plane and is the frontier of exactly two components of its complement.*

This follows from the reference under §4, p. 736, Lemma IV.

6. *Summary and General Discussion.* The theorems and corollaries of §§4 and 5 show that, if M is a bounded multi-coherent continuum which can be disconnected by the sum of two sub-continua, there are three possible cases: (1) when M can be disconnected by the sum of two continua of condensation; (2) when M can be disconnected by the sum of two sub-continua which are not continua of condensation;

* W. A. Wilson, *On the separation of the plane by irreducible continua*, this Bulletin, vol. 33, p. 739.

(3) when of any two sub-continua disconnecting M, one is a continuum of condensation, but not the other. Cases (1) and (2) are not mutually exclusive; they have been treated in §§4 and 5 and the results there obtained may be summarized as follows.

THEOREM. *Let M be a bounded multi-coherent continuum which can be disconnected by the sum of two sub-continua, both or neither of which are continua of condensation. Then M is the union of two continua whose divisor is the sum of two closed sets between which both continua are irreducible.*

COROLLARY. *Let M satisfy the hypotheses of the above theorem and lie in a plane. Then M is the common frontier of exactly two of its complementary domains.*

To make the third case definite, let A, but not B, be a continuum of condensation of M. If we use the notation of §5, this means that $A \subset H+K$ and $B-B\cdot(H+K)\neq 0$. Also both H and K are indecomposable, for otherwise we would have Case (2). There are two possibilities: $H\cdot K\cdot B=0$ and $H\cdot K\cdot B\neq 0$. If $H\cdot K\cdot B=0$, B contains a sub-continuum C which is irreducible between H and K and may contain $B-B\cdot(H+K)$. If so, $M=C+H+K$, C is indecomposable, and $C+H$ and K are both irreducible between $C\cdot K$ and $H\cdot K$. On the other hand, this is not the only possibility, as the following example shows. Let a, b, c, d, e, and f be six points. Let H be irreducible between each pair of the points a, b, and c; let K be irreducible between each pair of the points a, d, and e; let C be irreducible between each pair of the points b, d, and f; let D be irreducible between each pair of the points c, e, and f; let $B=C+D$; let $H\cdot K=a$, $C\cdot D=f$, $H\cdot B=b+c$, and $K\cdot B=d+e$; and let $M=H+K+B$. Then M is multi-coherent and can be disconnected by the sum of the continua a and B, but M cannot be expressed as the union of two continua having two closed sets in common, between which both are irreducible.

Now suppose that $H\cdot K\cdot B\neq 0$. Set $L=B-B\cdot(H+K)$; then $L=M-(H+K)$ and is connected. Let C be a sub-

continuum of B irreducible about $H \cdot K \cdot B$. Then $C \supset L$, as otherwise C would disconnect M. Thus M is the union of three continua, H, K, and C, where H and K are indecomposable and irreducible between $H \cdot K \cdot A$ and $H \cdot K \cdot B$, and C is irreducible about $H \cdot K \cdot B$ and contained in B, but it can be proved that M is not the union of two continua P and Q such that $P \cdot Q = \alpha + \beta$, where $\alpha \cdot \beta = 0$ and α and β are closed sets between which P and Q are irreducible.

That this case may exist is seen from the following example. Let a, b, c, and d be four points in the XY-plane in three-space; let H be a continuum irreducible between each pair of the points a, b, and c; let K be a continuum irreducible between each pair of the points a, b, and d; let H and K lie on opposite sides of the XY-plane except for these four points; and let C be a continuum lying in the XY-plane and irreducible between each pair of the points b, c, and d, but not containing a. Then $M = H + K + C$ is multi-coherent and can be disconnected by the sum of the two continua a and C; but it is not the union of two continua P and Q having the properties given in the last two lines of the previous paragraph.

7. *Generalization of Previous Theorems.* The properties of multi-coherent continua thus far obtained are readily extended. The following is the analog of §3.

THEOREM. *Let n be an integer greater than* 1 *and let M be a bounded multi-coherent continuum which cannot be disconnected by any set of $n-1$ or fewer sub-continua. Let $A_1, A_2, \cdots, A_n$ be proper sub-continua of M and let $\sum_1^n A_i$ disconnect M. Then $K = M - \sum_1^n A_i$ has exactly two components.*

PROOF. The theorem has already been proved for $n = 2$; let $n \geqq 3$. Now M cannot be irreducible between each pair of the sets $\{A_i\}$, for then K would be connected.* Let P be a proper sub-continuum of M irreducible between, say, A_1 and A_2; then $P - P \cdot (A_1 + A_2)$ is connected and has

* For M is irreducible between A_1 and $\sum_2^n A_i$ and the assumption that K is not connected leads at once to a contradiction.

limiting points on both A_1 and A_2. Thus for at least one component k of K, $\bar{k}\cdot A_1 \neq 0 \neq \bar{k}\cdot A_2$.

Suppose first that $\bar{k}\cdot A_i \neq 0$ for r values of i, say for $i \leqq r < n$, and that $\bar{k}\cdot A_i = 0$ for $r < i \leqq n$. Then $k+\sum_1^r A_i$ is a continuum not meeting $\sum_{r+1}^n A_i$, and

$$K-k = M-[k+\textstyle\sum_1^r A_i+\sum_{r+1}^n A_i]$$

is not void and is connected by hypothesis, since the set in the brackets is the sum of $n-r+1$ continua. Let

$$k' = K-k+\textstyle\sum_{r+1}^n A_i.$$

Now

$$M = k+k'+\textstyle\sum_1^r A_i;\ \bar{k}\cdot k' = \bar{k}\cdot(K-k)+\bar{k}\sum_{r+1}^n A_i = 0;$$

and

$$k\cdot\bar{k}' = k\cdot\overline{(K-k)}+k\cdot\textstyle\sum_{r+1}^n A_i = 0.$$

Hence $\sum_1^r A_i$ disconnects M, which is contrary to the hypothesis.

Now let $\bar{k}\cdot A_i \neq 0$ for every i. Then $k+\sum_1^n A_i$ is a continuum and, as before, $K-k = M-(k+\sum_1^n A_i)$ is connected. This proves the theorem.

8. THEOREM. *Let $n \geqq 3$ and let M be a bounded multi-coherent continuum which cannot be disconnected by any set of $n-1$ or fewer sub-continua. Let $A_1, A_2, \cdots, A_n$ be proper sub-continua of M and let $\sum_1^n A_i$ disconnect M. Then M is the union of $\sum_1^n A_i$ and two indecomposable continua H and K such that $H\cdot K \subset \sum_1^n A_i$ and both H and K are irreducible between each pair of the continua $\{A_i\}$.*

PROOF. By §7 there are two components h and k of $M-\sum_1^n A_i$. Set $H=\bar{h}$ and $K=\bar{k}$. Then $H\cdot A_i$ and $K\cdot A_i$ are not void for any value of i.

Let P be a sub-continuum of H irreducible between A_1 and A_2, for example. If $P\cdot A_i = 0$ for any i, say $i=3$, we have a contradiction, for then $(A_1+P+A_2)+A_3+\sum_4^n A_i$ disconnects M and contains not more than $n-1$ components, unless $P=H$, in which case M is disconnected by $A_1+A_2+\sum_4^n A_i$, again a contradiction.

Thus $P\cdot A_i \neq 0$ for every i. Now let $P' \subset P$ and be irre-

ducible between A_1 and A_3. By the above reasoning $P' \cdot A_3 \neq 0$. Hence $P' = P$. Hence P is irreducible between each pair of the continua $\{A_i\}$ and so is indecomposable.

Finally $P \supset h$, as otherwise $P + \sum_1^n A_i$ would disconnect M. Hence $P = H$ and the theorem is proved.

COROLLARY. *An n-coherent bounded continuum can be disconnected by n sub-continua, but not by fewer.*

9. THEOREM. *Let $n \geqq 3$ and let M be a bounded multi-coherent continuum which cannot be disconnected by any set of $n-1$ or fewer sub-continua. Let $A_1, A_2, \cdots, A_n$ be proper sub-continua of M, let $\sum_1^n A_i$ disconnect M, and let H and K be the two indecomposable sub-continua of M which are irreducible between each pair of the continua $\{A_i\}$. If every A_i is a continuum of condensation, $M = H + K$ and $H \cdot K = \sum_1^n \alpha_i$, where each α_i is a closed sub-set of A_i. Otherwise there is but one continuum A_i which is not a continuum of condensation and M may not be the union of two continua whose divisor is the sum of two closed sets between which both continua are irreducible.*

PROOF. If two of the continua $\{A_i\}$, say A_1 and A_2, were not continua of condensation of M, we would have a contradiction, since $H + K + \sum_3^n A_i$ would disconnect M. If every A_i is a continuum of condensation, M has the properties stated above by §8. If one of the continua $\{A_i\}$ is not a continuum of condensation, an example similar to that of §6 shows the truth of the last statement of the theorem.

COROLLARY. *Let M satisfy the hypotheses of the above theorem and lie in a plane. If every A_i is a continuum of condensation, M is the common frontier of exactly n components of its complement.*

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CONCERNING COLLECTIONS OF CUTTINGS OF CONNECTED POINT SETS*

BY G. T. WHYBURN

Two point sets are mutually separated if they are mutually exclusive and neither of them contains a limit point of the other. A point set M is said to be not connected or connected according as M is or is not the sum of two non-vacuous mutually separated point sets. A subset X of a connected point set M is said to be a cutting of M, or is said to cut M, provided the set of points $M-X$ is not connected; X is called an irreducible cutting of M provided X cuts M but no proper subset of X cuts M. If A and B are subsets of a connected point set M, the subset X of M is said to separate A and B in M, or to cut M between A and B, provided that $M-X$ is the sum of two mutually separated sets $M_a(X)$ and $M_b(X)$ containing A and B respectively.

In this paper the ordinary notation of point set theory will be employed, for example, $\overline{X}=X+X'$, where X' is the set of all limit points of the set X, $K \subset H$ means that the set K is a subset of the set H, $K \cdot H$ means the set of points common to K and H, $F(R)$ denotes the boundary of the set R, etc. In addition, if X is a cutting of a connected point set M then, unless otherwise stated, the equation $M-X=M_a(X)+M_b(X)$ is to be interpreted as meaning that $M-X$ is the sum of two mutually separated sets $M_a(X)$ and $M_b(X)$ which contain the sets A and B respectively.

1. *Cuttings of Connected Sets in General.* The point sets considered in this section are assumed to lie in a separable metric space which will be denoted by S. It is apparent from the proofs of some of the theorems, however, that they hold in even more general space.

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THEOREM 1. *Let G be any uncountable collection of mutually exclusive cuttings of a connected and separable* point set M. Then there exist two points A and B of M which are separated in M by uncountably many elements of G.*

PROOF. Since M is separable, it therefore contains a countable set of points D such that M is a subset of $\overline{D}$. As G is uncountable and D is countable, there exists an uncountable subcollection G_1 of G such that no element of G_1 contains a point of D. Now if X is any element of G_1, $M-X$ is the sum of two mutually separated sets $M_1(X)$ and $M_2(X)$. Let P_1 and P_2 be points of $M_1(X)$ and $M_2(X)$ respectively. Since P_1 is a limit point of D and is not a limit point of $M_2(X)$, then D is not a subset of $M_2(X)$. And since $D \cdot X = 0$, D is not a subset of $M_2(X)+X$. Therefore $M_1(X)$ contains at least one point $A(X)$ of D. Similarly, $M_2(X)$ contains at least one point $B(X)$ of D. Thus every element X of G_1 separates some pair of points $A(X)$ and $B(X)$ of D in M. And since the elements of G_1 are uncountable and the collection of all pairs of points of D is countable, it follows that some two points A and B of D are separated in M by uncountably many elements of G_1.

THEOREM 2. *If G_0 is any collection of mutually exclusive subsets of a connected and separable set M each of which contains a cutting of M, then all save possibly a countable number of the elements of G_0 must be cuttings of M.*

PROOF. Since M is separable, it contains a countable set of points D such that $M \subset \overline{D}$. By hypothesis each element g of G_0 contains a cutting X_g of M. Now let G be the collection of all those elements of G_0 which are not cuttings

* It follows by a theorem of W. Gross (See *Zur Theorie der Mengen, in denen ein Distanzbegriff definiert ist*, Wiener Sitzungsberichte, vol. 123 (1914), pp. 801–819) that every subset of a separable metric space is separable. Hence that M is separable follows from the fact that it is imbedded in the space S. However, the condition that M be separable is explicitly stated in this theorem and in some of the theorems that are to follow because the separability of M is used in the proof and because as thus stated the theorem holds in a more general space than the space S.

of M. For each element g of G, $M-X_g$ is the sum of two mutually separated sets M_1 and M_2; and since $M-g$ is connected and a subset of $M-X_g$, it therefore is a subset either of M_1 or of M_2, say of M_1. Then obviously M_2 is a subset of g. Let P be a point of M_2. Then since P is a limit point of D but not of M_1, D is not a subset of M_1. Hence M_2+X_g contains at least one point of D; and as M_2+X_g is a subset of g, then g contains at least one point of D. Thus every element of G contains at least one point of D. And since the elements of G are mutually exclusive and D is countable, it follows that G is countable.

THEOREM 3. *If X is any cutting of a connected set M between two points A and B of M, and K is any subset of M which contains neither A nor B, then $X+K$ cuts M between A and B.*

PROOF. By hypothesis $M-X=M_a(X)+M_b(X)$. Then $M-(X+K)=[M_a(X)-M_a(X)\cdot K]+[M_b(X)-M_b(X)\cdot K]$, and clearly $[M_a(X)-M_a(X)\cdot K]$ and $[M_b(X)-M_b(X)\cdot K]$ are mutually separated and contain A and B respectively.

THEOREM 4. If *N is any connected subset of a connected and separable set M, then not more than a countable number of the components* of $M-N$ can contain cuttings of M.*

PROOF. By a theorem of Knaster and Kuratowski's,† no component of $M-N$ can cut M. Therefore by Theorem 2, not more than a countable number of the components of $M-N$ can contain any subset which cuts M.

THEOREM 5. *If G_0 is a collection of mutually exclusive connected subsets of a connected and separable set M such that for each element g of G_0, $M-g$ is neither connected nor the sum of two connected sets, then G_0 is countable.*‡

* A component of a point set K is a connected subset of K which is not a proper subset of any connected subset of K. See Hausdorff, *Mengenlehre*.

† *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 206–253, Theorem 10.

‡ This theorem is a generalization of a theorem due to Kuratowski and Zarankiewicz (see *A theorem on connected point sets*, this Bulletin, vol. 33

PROOF. Suppose, on the contrary, that G_0 is uncountable. Then by Theorem 1, there exist two points A and B of M and an uncountable subcollection G of G_0 each element of which separates A and B in M. Since M is separable, it contains a countable set of points D such that $M \subset \overline{D}$. For each element g of G, $M-g$ is the sum of three mutually separated sets M_a, M_b, and M_g, where M_a and M_b contain A and B respectively. For each element g of G, it follows as in the preceding proofs that M_g+g contains at least one point of D. And since,* for each g, M_a+M_b+g is connected, and contains both A and B, it therefore contains every element of G. Hence for every two distinct elements X and Y of G, M_x+X and M_y+Y are mutually exclusive sets; and as each such set contains a point of D, and D is countable, it follows that G is countable, contrary to supposition. Thus the supposition that Theorem 5 is false leads to a contradiction.

Let A and B be two connected subsets (or points) of a connected point set M, and let G be any collection of mutually exclusive connected subsets of M each of which separates A and B in M. Then the elements of G can be linearly ordered in M from A to B as follows. The element X of G will be said to precede or to follow the element Y of G in M in the order from A to B according as X belongs to $M_a(Y)$ or to $M_b(Y)$, where $M_a(Y)+M_b(Y)$ is one method of expressing $M-Y$ as the sum of two mutually separated sets containing A and B respectively. It is readily deduced from this definition that (a) of two elements X and Y of G, one must always precede the other, (b) X cannot both precede and follow Y, and (c) if X precedes Y, then Y follows X.

(1927), p. 574), who assume the unnecessary condition that the elements of G_0 are closed relative to M. Although the proof here given for Theorem 5 differs markedly from that given by Kuratowski and Zarankiewicz to prove a special case of their theorem, I have found that a proof can be constructed for Theorem 5 based on their methods which is somewhat shorter than the proof they gave in the paper just mentioned and which makes no use of the fact that the set M itself is separable.

* See Knaster and Kuratowski, loc. cit., Theorem 6.

If X and Y are distinct elements of G and X precedes Y in M in the order from A to B, then by the segment $S(X, Y)$ of M from X to Y is meant the set of points $M_b(X) \cdot M_b(Y)$, where $M_a(X)+M_b(X)$ is one way of expressing $M-X$ as the sum of two mutually separated sets containing A and B respectively and similarly for $M_a(Y)$ and $M_b(Y)$; by the interval $I(X, Y)$ of M from X to Y is meant the set of points $[M_b(X)+X] \cdot [M_a(Y)+Y]$. Clearly $I(X, Y)=S(X, Y)+X+Y$. In case all of the sets $M_a(X)$, $M_b(X)$, $M_a(Y)$, and $M_b(Y)$ are connected, as is necessarily the case for all save possibly a countable number of elements of G (see Theorem 5), then $S(X, Y)$ and $I(X, Y)$ are uniquely determined. The following facts are readily proved.

(i). For each pair of elements X and Y of G, where X precedes Y, M is the sum of three connected point sets $M_a(X)+X$, $I(X, Y)$, and $M_b(Y)+Y$, where we have $[M_a(X)+X] \cdot I(X, Y)=X$, $[M_b(Y)+Y] \cdot I(X, Y)=Y$, and $[M_a(X)+X] \cdot [M_b(Y)+Y]=0$.

(ii). No point of $S(X, Y)$ is a limit point of $M-I(X, Y)$.

(iii). If $M_a(X)$, $M_b(X)$, $M_a(Y)$, and $M_b(Y)$ are connected, and $S(X, Y)$ contains an element of G, then $S(X, Y)$ is connected. It may happen that for some element X of G, an element Y of G exists such that $S(X, Y)$ is vacuous. However, Theorem 6 below shows that this can be the case for at most a countable number of elements X of G.

THEOREM 6. *Let G be any uncountable collection of mutually exclusive connected cuttings of a connected set M. Then there exists a subcollection of G^* of G, containing all but a countable number of the elements of G, and such that every two elements of G^* are separated in M by uncountably many elements of G.**

PROOF. Since† M is separable, it contains a countable subset D such that $M \subset \overline{D}$. Let G_1 denote the collection obtained by omitting from the collection G (1) all elements X of G such that $M-X$ is not the sum of two connected point sets, (2) all elements of G which contain at least one

† See Gross, loc. cit.

point of D, and (3) all elements X of G which separate in M some two points of D which are separated in M by only a countable number of elements of G. Then G_1 contains all but a countable number of the elements of G. Let H denote the collection of all possible pairs of points of the set D having the property that each pair A, B in H is separated in M by each element of an uncountable subcollection G_{ab} of the collection G_1. For each pair A, B in H, let E_{ab} denote the collection of all those elements e of G_{ab} such that every neighborhood of e contains points of uncountably many elements of G_{ab}. By a theorem due to Gross (loc. cit.), E_{ab} contains all but a countable number of the elements of G_{ab}. Hence every neighborhood of each element e of E_{ab} contains points of uncountably many elements of E_{ab}.

Now for each pair of elements X and Y of E_{ab}, $S(X, Y)$ is non-vacuous. For suppose $S(X, Y)$ is vacuous, and suppose X precedes Y in M in the order from A to B. Now X contains a point P which is a limit point of $M_b(X)$. And since P is not a limit point of $M_b(Y)$ [for X belongs to $M_a(Y)$], there exists a neighborhood V of P which contains no point of $M_b(Y)$. Now V contains at least one point Q of $M_b(X)$; and since $M_b(X) = M_b(Y) + Y + S(X, Y)$, and $S(X, Y)$ is vacuous, then Q must belong to Y. Now Q is not a limit point of $M_a(X)$, for Q belongs to Y and Y belongs to $M_b(X)$. Hence there exists a neighborhood U of Q which contains no point of $M_a(X)$ and which is a subset of V. Then $U \cdot M$ is a subset of $X+Y$. But $U \cdot M$ must contain at least one point of D; and since X and Y belong to G_1, neither of them can contain a point of D. Thus the supposition that $S(X, Y)$ is vacuous leads to a contradiction.

Now the number of elements X of E_{ab} such that an element Y of E_{ab} exists such that $S(X, Y)$ contains no element of E_{ab} must be countable. For each such segment $S(X, Y)$ must contain at least one point P; and since P is a limit point of D but not of $M - I(X, Y)$, and neither X or Y contains a point of D, then $S(X, Y)$ must contain at least one point of D. And clearly if $S(X_1, Y_1)$ and $S(X_2, Y_2)$ are two

such segments which are different, they must be mutually exclusive. Therefore the number of such segments must be countable, and hence all save a countable number of elements X of E_{ab} must have the property that if Y is any other element whatever of E_{ab}, then $S(X, Y)$ contains at least one element Z of E_{ab}. Let G^*_{ab} denote the collection of all those elements of E_{ab} which have this property. Then if X and Y are any two elements of G^*_{ab}, $S(X, Y)$ contains at least one element Z of E; and since every neighborhood of Z contains points of uncountably many elements of E_{ab}, since no point of Z is a limit point of $M-I(X, Y)$, and since G^*_{ab} contains all but a countable number of the elements of E_{ab}, it follows that $S(X, Y)$ contains uncountably many elements of G^*_{ab}.

Let G^* denote the collection of all those elements g of G such that g belongs to some collection G^*_{ab}. Since H is countable, it follows that G^* contains all save a countable number of the elements of G. Now let X and Y be any two distinct elements of G^*. Then $M-X=M_y+M_1$, where M_y and M_1 are mutually separated and M_y contains Y. It is readily seen that M_1 contains at least one point A of D. Likewise $M-Y=M_x+M_2$, and M_2 contains a point B of D. It is readily seen that both X and Y separate A and B in M. Since X and Y belong to G_1, then A and B are separated in M by uncountably many elements of G. Hence A,B is a pair in H, and X and Y belong to G^*_{ab}. Therefore $S(X, Y)$ contains uncountably many elements $[Z]$ of G^*_{ab}. Since† $M_a(X)+X$ and $M_b(Y)+Y$ are connected and contain A and B respectively but contain no point of Z, for each Z in $[Z]$, then X belongs to $M_a(Z)$ and Y to $M_b(Z)$. Hence Z separates X and Y in M. Thus every two elements X and Y of G^* are separated in M by uncountably many elements of G,* and our theorem is proved.

THEOREM 7. *If G^* is the collection obtained in Theorem 6 and E is the collection of elements e such that $e=\bar{g}\cdot M$, for*

† See Knaster and Kuratowski, loc. cit., Theorem 6.

each element g of G, then the elements of E are mutually exclusive, all save a countable number are cuttings of M and every two of them are separated in M by some third element of E (indeed, by uncountably many elements of E).*

PROOF. Let d and e be two distinct elements of E, where $d=\bar{g}\cdot M$ and $e=\bar{h}\cdot M$, where g and h are elements of G^*. There exists an element i of G^* such that $M-i=M_g+M_h$. Hence h contains no limit point of g; similarly it is shown that i contains no limit point of either g or h. Then since d and e are subsets of M_g+i and M_h+i respectively, and $d\cdot i=e\cdot i=0$, it follows that d and e are subsets of M_g and M_h respectively. Hence d and e are mutually exclusive. A similar argument shows that if f is the element of E such that $f=\bar{i}\cdot M$, then f separates d and e in M. That all save possibly a countable number of the elements of E are cuttings of M follows now from Theorem 2. The truth of Theorem 7 is therefore established.

2. *Cuttings of Continua.* In this section the point sets considered are assumed to lie in a locally compact, metric, and separable space, that is, the same space considered in §1 with the additional restriction that it be locally compact.

THEOREM 8. *No continuum M contains an uncountable collection G of mutually exclusive connected subsets each of which contains a compact proper subset which cuts M.*†

PROOF. Suppose, on the contrary, that some continuum M contains such a collection G. Each element g of G contains a closed‡ and compact set f which cuts M and a point

† For the special case where the continuum M is identical with a whole euclidean space of n dimensions, this theorem is readily deduced from a theorem of Kuratowski's; see C. Kuratowski, *Sur les coupures irréductibles du plan*, Fundamenta Mathematicae, vol. 6 (1924), pp. 130–146, Theorem 8.

‡ For justification of the word "closed" here see R. L. Moore (Proceedings of the National Academy of Sciences, vol. 10 (1924), pp. 356–360), who proves, with the aid of a theorem of Knaster and Kuratowski (*Sur les ensembles connexes*, loc. cit., Theorem 37), that every cutting of a bounded plane continuum M between two points A and B of M contains a closed cutting of M between A and B. Although the theorem of Knaster

p not belonging to f. For each element g of G select two such sets f and p, let F denote the collection of sets $[f]$, and let P denote the set of points $[p]$. For each element g of G, let $\delta(f, p)$ be the minimum distance between the sets f and p. There exists a number $\epsilon>0$ and an uncountable subcollection G_1 of G such that for each element g_1 of G_1, $\delta(f_1, p_1) > 4\epsilon$. Let F_1 and P_1 denote the collections $[f_1]$ and $[p_1]$, for all elements g_1 of G_1. There exists a point p which is a point of condensation of P_1. Let R be the set of all points whose distance from p is less than ϵ. Then no element of F_1 contains a point of $\overline{R}$. Let P_2 be the set of points $R \cdot P_1$, and let F_2 and G_2 be the corresponding subcollections of F_1 and G_1 respectively, that is, each point p_2 of P_2 corresponds to an element f_2 in F_2 and an element g_2 in G_2. By a theorem* established in its most general form by R. L. Moore, there exists an element f of F_2 such that every neighborhood of f contains uncountably many elements of F_2. Since f is compact, it follows with the aid of the Borel theorem that there exists a compact open set D which contains f but contains no point whatever of R. Let F_3 be the collection of all those elements of F_2 which lie wholly in D, and let G_3 be the corresponding collection of elements of G_2.

By Theorems 1, 3, and 6 it follows that there exist two points A and B of M and an uncountable subcollection G_4 of G_3 such that if F_4 is the corresponding collection of elements of F, then (1) each element of F_4 and also each element of G_4 separates A and B in M, and (2) every two elements X and Y of G_4 are separated in M by uncountably

and Kuratowski used by Moore does not necessarily remain true in the space considered in this paper, nevertheless the methods of proof used by Knaster and Kuratowski in proving their theorem suffice to prove Moore's theorem in such a space, and indeed to prove the following more general theorem: *If K is a cutting of a continuum M (bounded or not) between any two subsets A and B of M, then K contains a closed cutting of M between A and B.*

* See R. L. Moore, *Concerning triodic continua in the plane*, to appear in Fundamenta Mathematicae; for the case of a euclidean space see a theorem of C. Zarankiewicz in his paper *Sur les points de division dans les ensembles connexes*, Fundamenta Mathematicae, vol. 9 (1927), Theorem 2.

many elements of G^4. Let T denote the point set consisting of the sum of all the point sets of the collection G_4, and for each element g of G_4, let e denote the component of $M-(T-g)$ which contains g, and let E denote the collection of sets $[e]$ thus obtained. Then each element e of E must be closed; for if some element h of G_4, different from the element of G_4 which belongs to e, contained a limit point of g, then $e+h$ would be connected and would contain both e and h, contrary to the fact that an element j of G_4 exists which contains no point of $e+h$ but which separates g and h in M. Hence each element of E is a continuum, and a similar argument shows that the elements of E are mutually exclusive and that every two elements of E are separated in M by some third element of E. Since E is uncountable, it follows readily from Theorems 3 and 5 together with the theorem of Moore-Zarankiewicz quoted above that there exists an element e of E such that (1) e separates A and B in M and both $M_a(e)$ and $M_b(e)$ are connected, (2) either every point of e is a limit point of $M_a(e)$ or every point of e is a limit point of $M_b(e)$, suppose the former, and (3) there exists a point p of e and a sequence of points $p_1, p_2, p_0, \cdots$, such that for each i, p_i belongs to an element e_i of E which belongs to $M_b(e)$, and such that p is the sequential limit point of the sequence of points $p_1, p_2, p_3, \cdots$. For each i, let q_i denote the component of $e_i \cdot \overline{D}$ which contains p_i, and let K denote the sequential limiting set of the sequence $q_1, q_2, q_3, \cdots$. By a theorem due to Janiszewski,* for each i, q_i is a compact continuum containing at least one point of $F(D)$. Hence by a theorem proved in its most general form by R. G. Lubben,† it follows that K is a continuum. Clearly K contains at least one point X of $F(D)$. Now K contains no point of $T-e$. For suppose K does contain a point Y of $T-e$; then Y belongs to some element j of E; and

* *Sur les continus irréductibles entre deux points*, Journal de l'École Polytechnique, (2), vol. 16 (1912).

† See R. G. Lubben, *Concerning limiting sets in abstract spaces*, Transactions of this Society, vol. 30 (1928), pp. 668–685.

there exists an element d of E such that $M-d=M_e(d)+M_j(e)$. Either $M_e(d)$ or $M_j(e)$ must contain infinitely many elements of the sequence $q_1, q_2, q_3, \cdots$. But $M_e(d)$ cannot contain infinitely many elements of this sequence; for if it did, then Y, a point of $M_j(d)$, would be a limit point of $M_e(d)$. Likewise $M_j(d)$ cannot contain infinitely many elements of this sequence; for if so, then p, a point of $M_e(d)$, would be a limit point of $M_j(d)$. Thus the supposition that K contains a point of $T-e$ leads to a contradiction. Then since K is a connected subset of M which contains p but contains no point of $T-e$, and e is the component of $M-(T-e)$ which contains p, K must be a subset of e. Hence e contains the point X. Then X is a limit point of $M_a(e)$, for every point of e is a limit point of $M_a(e)$; and X is also a limit point of $M_b(e)$. Hence $M_a(e)+X+M_b(e)$ is connected and contains both A and B. But this is contrary to the fact that e contains an element f of F which lies wholly in D (and hence does not contain X) and which separates A and B in M. Thus the supposition that Theorem 8 is false leads to a contradiction.

COROLLARY. *If G is any collection of mutually exclusive connected subsets of a continuum M each of which contains a compact cutting of M, then all save possibly a countable number of the elements of G are themselves compact continua which are irreducible cuttings of M.*

THEOREM 9. *If G is any collection of mutually exclusive connected subsets of a continuum M each of which contains a compact cutting of M, and T denotes the point set obtained by adding together all the point sets of the collection G, then all save possibly a countable number of the elements of G are components of $M-(T-g)$; indeed, all save a countable number of the elements g of G have the property that every connected subset of M which contains g but is not identical with g must contain points of uncountably many elements of G.*

PROOF. By Theorems 6 and 8, G contains a subcollection G_1 which contains all but a countable number of the elements

of G and such that the elements of G_1 are compact continua which are irreducible cuttings of M and have the property that every two of them are separated in M by uncountably many elements of G_1. Now let E denote the collection of all those elements g of G_1 such that g is a proper subset of some connected subset X of M which has points in common with at most a countable number of elements of G. Let $[X]$ denote the collection of sets X. Then since every two elements of G_1 are separated in M by uncountably many elements of G_1, it follows that no element of $[X]$ contains points of more than one element of G_1 and that no two elements of $[X]$ have a point in common. And since each element of $[X]$ contains a compact proper subset which cuts M, it follows by Theorem 8 that $[X]$ must be countable. Therefore E is countable.

THEOREM 10. *If G_0 is any uncountable collection of mutually exclusive connected subsets of a continuum M each of which contains a compact cutting of M, then G_0 contains a subcollection G which contains all save possibly a countable number of the elements of G_0 and has the following properties*: (1) *every element g of G is a compact continuum which is an irreducible cutting of M and is a component of the set of points $M-(T-g)$, where T denotes the point set obtained by adding together all the point sets of the collection G,* (2) *every two elements of G are separated in M by uncountably many elements of G, and* (3) *G is upper semi-continuous.**

PROOF. That G_0 contains a collection G having properties (1) and (2) follows immediately from Theorems 6, 8, and 9. I shall now show that any collection G of mutually exclusive

* A collection G of point sets is said to be upper semi-continuous provided that if g is any element of G and $g_1, g_2, g_3, \cdots$ is any infinite sequence of elements of G containing points $P_1, P_2, \cdots$, respectively, such that the sequence of points $P_1, P_2, \cdots$ has a sequential limit point which belongs to g, then g contains the entire sequential limiting set of the sequence of sets $g_1, g_2, g_3, \cdots$. See R. L. Moore, *Concerning upper semi-continuous collections of continua*, Transactions of this Society, vol. 27 (1925), pp. 416–428.

cuttings of M having properties (1) and (2) must be upper semi-continuous. Let $g_1, g_2, g_3, \cdots$ be any sequence of elements of G containing points $P_1, P_2, P_3, \cdots$, respectively, such that the sequence $P_1, P_2, P_3, \cdots$ has a sequential limit point P belonging to some element g of G. Let L denote the sequential limiting set of the sequence $g_1, g_2, \cdots$. Then g must contain L. For if not, then since g is compact, it follows readily that g is a proper subset of the component K of $g+L$ which contains g. And, since g is a component of $M-(T-g)$, K must contain a point Q of some other element h of G different from g. And by (2) there exists an element f of G which separates g and h in M, that is, $M-f=M_g(f)+M_h(f)$. Now either $M_g(f)$ or $M_h(f)$ must contain infinitely many sets of the sequence $g_1, g_2, \cdots$. But it is easy to see that neither of these sets can contain infinitely many sets of this sequence, because each of these sets contains a point of L. Hence g contains L, and therefore G is an upper semi-continuous collection.

COROLLARY. *If G is any collection of mutually exclusive compact subcontinua of a continuum M such that* (1) *each element g of G is saturated with respect to the property of being a subcontinuum of $M-(T-g)$, where T is the point set obtained by adding together all the point sets of the collection G, and* (2) *every two elements of G are separated in M by some third element of G, then G is upper semi-continuous.*

Suppose M is a compact continuum and G is a collection of mutually exclusive sets having properties (1) and (2) in the statement of Theorem 10. Then by property (1) it follows that every component of $M-T$ is closed and hence is a compact continuum. And if G_0 denotes the collection whose elements are the elements of G together with all the components of $M-T$, then the sum of all the elements of G_0 is identical with M; and using property (2) of G it is readily shown that M is an acyclic continuous curve (that is, a connected im kleinen continuum which contains no simple closed curve) with respect to the elements of G_0, and thus G_0 is upper semi-continuous. The same is true if instead of as-

suming M compact we assume merely that each component of $M-T$ is compact. Thus we have the following theorem.

THEOREM 11. *If M is any compact continuum and G is any uncountable collection of mutually exclusive connected subsets of M each of which contains a cutting of M, then there exists an upper semi-continuous collection G_0 of mutually exclusive compact subcontinua of M such that* (1) *all save possibly a countable number of elements of G are elements also of G_0,* (2) *the sum of all the elements of G_0 is identical with M, and* (3) *M is an acyclic continuous curve with respect to the elements of G_0.*

3. *The Order of Various Cuttings and im Kleinen Cuttings of Continua.** With the aid of the results established above it follows that all of the theorems, except Theorem 10, in the author's paper *Concerning the cut points of continua*† hold true in any separable metric space which is locally compact. In particular, the theorem (loc. cit., Theorem 7) that *all save possibly a countable number of the cut points of any continuum M are points of Menger order two of M* holds true in such a space. Using the results proved above and a method essentially the same as that used to prove Theorem 7 in the paper of the author's just mentioned, it is not difficult to prove the following more general theorem.

THEOREM 12. *If M is any continuum and G is any collection of mutually exclusive compact subcontinua of M each of which cuts M, then all, save possibly a countable number, elements X of G have the property that for each $\epsilon>0$, an open set R exists containing X and every point of which is at a distance $<\epsilon$ from some point of X and such that $F(R)\cdot M=Y+Z$, where Y and Z are continua of the collection G.*

Thus all save possibly a countable number of elements of G are continua of Menger order (or Urysohn index) two of

* The space considered in this section is the same as the one considered in §2.

† Transactions of this Society, vol. 30 (1928), No. 3, pp. 597–609.

M in a certain generalized sense. Menger* and Urysohn* call a point P of a continuum M a point of "Verzweigungs" order (or index) n of M provided that for each $\epsilon>0$, P can be ϵ-separated† by n subcontinua of M but not by $n-1$ such continua, that is, for each $\epsilon>0$ an open set R exists which contains P and is of diameter $<\epsilon$ and such that $F(R)\cdot M$ contains at most n components and such that n is the least integer such that this property is preserved. If we extend this notion to include "continua P" as well as "points P," and say that a subcontinuum P of a continuum M is ϵ-separated by n subcontinua of M provided that for each $\epsilon>0$ an open set R exists which contains P and such that (1) every point of R is at a distance $<\epsilon$ from some point of P and (2) $F(R)\cdot M$ has at most n components, then Theorem 12 states that all save possibly a countable number of the elements of G are continua of "Verzweigungs" order two of M, or indeed, that all save possibly a countable number of the elements of G can, for each $\epsilon>0$, be ϵ-separated by two continua *which are elements of G.*

If C is a subcontinuum of a continuum M such that for each $\epsilon>0$, an open set R exists which contains C and such that (1) every point of R is at a distance $<\epsilon$ from some point of C and (2) C is a cutting of the component of $M\cdot\overline{R}$ which contains C, then C will be called an *im kleinen cut continuum* of M. For the special case where C is a single point, C will be called an *im kleinen cut point*, or a *cut point im kleinen*, of M. The notion of a cut point im kleinen of a continuum is embodied in Urysohn's notion of an "unvermeidbar" (unavoidable) point of a continuum‡ and in R. L. Moore's

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* See K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), pp. 276–306, and P. Urysohn, *Über im kleinen zusammenhängende Kontinua*, Mathematische Annalen, vol. 98 (1927), pp. 296–308, and earlier papers by the same authors referred to therein.

† See P. Urysohn, *Sur la ramification des lignes Cantoriennes*, Comptes Rendus, vol. 175 (1922), p. 481.

‡ See P. Urysohn, *Über im kleinen zusammenhängende Kontinua*, loc. cit.; and R. L. Moore, *Concerning triods in the plane and the junction points*

notion of a "junction" point of a continuum. For the case of a continuous curve (that is, a connected im kleinen continuum) it has been observed by W. L. Ayres and the author that the notions of an unavoidable point in the sense of Urysohn and of an im kleinen cut point as above defined are equivalent.

THEOREM 13. *If H denotes the set of all the points of a continuum M at which M is connected im kleinen and K denotes the set of all the im kleinen cut points of M, then all save possibly a countable number of the points of $K \cdot H$ are points of Menger order two of M.*

PROOF. Suppose, on the contrary, that $H \cdot K$ contains an uncountable subset N no point of which is a point of order two of M. For each point P of N there exists a number $E_p > 0$ such that if R_p denotes the component of $M \cdot S(P, E_p)$ containing P, where $S(P, E_p)$ denotes the set of all points whose distance from P is $\leqq E_p$, then P is a cut point of R_p. There exists a number $e > 0$ and an uncountable subset N_1 of N such that for each point P_1 of N_1, $E_{p_1} > 4e$. There exists a point Q of N_1 which is a point of condensation of N_1. Let W denote the component of $M \cdot S(Q, e)$ which contains Q. Since M is connected im kleinen at Q and Q is a point of condensation of N_1, there exists an uncountable subset N_2 of N_1 every point of which belongs to W and is at a distance $< e/2$ from Q. Then for each point X of N_2, R_x must contain W, because $E_x > 4e$. Hence, by a lemma of R. L. Moore's,* each point of N_2 is a cut point of W. Then by the theorem of the author's stated in the first paragraph of this section it follows that there exists at least one point Y of N_2 which is a point of Menger order two of W. And since Y is not a limit point of $M - W$ (for M is connected im kleinen at Y), then Y is a point of order two of M, contrary to hypothesis. Thus the supposition that Theorem 13 is false leads to a contradiction.

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of plane continua, Proceedings of the National Academy of Sciences, vol. 14 (1928), pp. 85–88.

* *Concerning triods* · · · , loc. cit., Lemma 2.

COROLLARY 1. *All save possibly a countable number of the cut points im kleinen of any continuous curve M are points of Menger order two of M.*

COROLLARY 2. *The junction points* of any continuous curve are countable.*

THEOREM 14. *If every subcontinuum of a continuous curve M contains uncountably many im kleinen cut points of M, then M is a Menger regular curve.*†

Theorem 14 follows at once from Theorem 13 and Menger's theorem (loc. cit., Theorem 8) that the set of all non-regular points of any continuum either is vacuous or else contains a continuum. For the case where M lies in the plane, Theorem 13 is related to Theorem 11 in the author's paper *Concerning certain accessible points of plane continua.*‡

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* A junction point of a continuum M is a cut point im kleinen of M which is the emanation point of some triod which lies in M, where by a triod is meant a continuum which is the sum of three continua AO, BO and CO, irreducible between the points A and O, B and O, and C and O respectively, and such that O is the only point common to any two of them, and by the emanation point of a triod is meant the point corresponding to the point O in the definition just given. (See R. L. Moore, *Concerning triods* · · · , loc. cit.) Moore shows that the junction points of any *plane* continuum are countable, whether M is a continuous curve or not. That Moore's theorem is not true in 3-space is shown by the following example. Let K be a non-dense perfect set on an interval I of the X-axis. For each point X of K, with X-coordinate x, let T_x denote the triod which is the sum of the straight line interval joining the points $(x, 0, 0)$ and $(x, 2, 0)$ and the one joining $(x, 1, 0)$ and $(x, 1, 1)$. Let $M = I + \sum_{x \subset K} T_x$. Then for each point X of K, the point $(x, 1, 0)$ is a junction point of the continuum M. Corollary 2 shows that the condition in Moore's theorem that the continuum M lie in the plane may be replaced by the condition that M be a continuous curve. Incidentally, Corollary 2 gives a more general result than the results of Wazewski-Menger and Alexandroff referred to in the introduction of the above mentioned paper of Moore's, even in n dimensions or in any locally compact, metric, and separable space.

† A Menger regular curve is a continuum M all of whose points are regular in the sense of Menger, that is, each point of M can, for each $\epsilon > 0$, be ϵ-separated by a finite number of points of M; see K. Menger, *Grundzüge einer Theorie der Kurven*, loc. cit.

‡ Monatshefte für Mathematik und Physik, vol. 35 (1928), pp. 289–304.

COROLLARY. *If every point of a continuous curve M is a cut point im kleinen of M, (or is an unavoidable point of M in the sense of Urysohn), then M is a Menger regular curve.**

A continuous curve having the property stated in this corollary, that is, every one of its points is an im kleinen cut point, has a number of interesting simple properties, such as, for example, every subcontinuum of any one of its maximal cyclic curves C contains an arc segment which is an open subset of C.

Using the theorems in the preceding sections together with methods similar to those used in the proof of Theorem 13, it follows that *if G is any collection of mutually exclusive compact im kleinen cut continua of a continuous curve M, then all save possibly a countable number of the continua of G are continua of "Verzweigungs" order two of M.* Likewise if we follow R. L. Moore (loc. cit.) and call a subcontinuum N of a continuum M a *junction continuum* of M provided N is an im kleinen cut continuum of M and is the emanation continuum of some *analog of a triod*† belonging to M, then *no continuous curve contains an uncountable collection of mutually exclusive compact junction continua of itself.*

For the case of the plane, it follows from the results of this paper by methods of proof essentially the same as used by R. L. Moore in proving Theorem 5 of his paper *Concerning triods in the plane and the junction points of plane continua* (loc. cit.) that *no plane continuum contains an uncountable collection of mutually exclusive compact junction continua of itself.*

THE UNIVERSITY OF TEXAS

* For the case where M lies in the plane, this corollary is closely related to Theorem 2 of my paper *Concerning accessibility in the plane and regular accessibility in n dimensions*, this Bulletin, vol. 34 (1928), pp. 504-510.

† See R. L. Moore, *Concerning triodic continua in the plane*, to appear in Fundamenta Mathematicae. By an *analog of a triod* is meant a continuum which differs from a triod only in that it emanates from a continuum instead of from a point.

APPROXIMATE SOLUTIONS OF CERTAIN GENERAL TYPES OF BOUNDARY PROBLEMS FROM THE STANDPOINT OF INTEGRAL EQUATIONS*

BY ROSE WHELAN

1. *Introduction.* The principle of Rayleigh†, which is used by physicists to obtain practical solutions of certain boundary problems, assumes that the system under consideration may be replaced by an approximating algebraic system so chosen that the difference between the solutions of the two systems is negligible. It is evident that the practical value of this principle depends not only on the possibility of choosing an approximating system which can be easily solved, but also on the possibility of estimating the order of difference between the solutions.

In a paper published in 1923, M. Plancherel‡ presented a justification of the use of the difference system as an approximating system for a second order self adjoint linear differential equation with simple boundary conditions. He does not discuss the order of approximation of solutions. Such a discussion has been given by N. Bogoliouboff and N. Kryloff.§ R. Courant,‖ in a paper which appeared in 1926, showed that certain integro-differential boundary problems could also be approximated by the method of

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* Presented to the Society, September 7, 1928. This paper is an abbreviated version of a thesis submitted to Brown University in candidacy for the degree of Doctor of Philosophy.

† Lord Rayleigh, *Theory of Sound*, vol. 1, pp. 89–96.

‡ M. Plancherel, Bulletin des Sciences Mathématiques, (2), vol. 47 (1923), pp. 153-160, 170–177.

§ N. Bogoliouboff and N. Kryloff, Annals of Mathematics, (2), vol. 29 (1928), pp. 255–275.

‖ R. Courant, Acta Mathematica, vol. 49 (1926), pp. 1-67.

difference equations. In this paper we use an essentially different method.* The class of problems to which it can be applied is not restricted to the class of problems that originated in the calculus of variations, with which Courant has been concerned. In some respects, however, the method used by Courant is more general since it can be applied to non-linear as well as to partial differential boundary problems.

2. *Preliminary Theorems: Approximate Solutions of the Fredholm Integral Equation.*† We obtain for reference in this and later sections an estimate of the differences between the characteristic numbers of the Fredholm integral equation

$$(A_1) \qquad u(x) + \lambda \int_a^b k(x, s)u(s)ds = f(x),$$

and those of

$$(A_2) \qquad u(x,\epsilon) + \lambda \int_a^b k(x,s,\epsilon)u(s,\epsilon)ds = f(x,\epsilon),$$

where, in the region $a \leqq \frac{x}{s} \leqq b$, $0 \leqq \epsilon \leqq \epsilon_0$, the functions $k(x, s, \epsilon)$ and $k(x, s)$ are bounded and integrable (in the Lebesgue sense) and

$$k(x,s,\epsilon) - k(x,s) = O(\epsilon)\,; \quad f(x,\epsilon) - f(x) = O(\epsilon).\ddagger$$

If ϵ_0 is sufficiently small, then

$$k(x,s,\epsilon) - k(x,s) = \epsilon h(x,s,\epsilon)$$

where $h(x, s, \epsilon)$ is uniformly bounded in the region $a \leqq \frac{x}{s} \leqq b$, $0 \leqq \epsilon \leqq \epsilon_0$, and $f(x, \epsilon) - f(x) = \epsilon h(x, \epsilon)$, where $h(x, \epsilon)$ is uniformly

* This method was suggested by Professor J. D. Tamarkin to whom the author is indebted for advice and criticism during the preparation of this paper.

† E. Schmidt, Mathematische Annalen, vol. 63, pp. 467–472, and vol. 64, pp. 161–174; H. Bateman, Proceedings of the Royal Society, A, vol. 100, pp. 441–449; F. Tricomi, R. Accademia dei Lincei, vol. 33, sem. 1, pp. 483–486, and vol. 33, sem. 2, pp. 26–30; R. B. Adams, Thesis in Candidacy for the degree of Doctor of Philosophy submitted to Radcliffe College, 1921; H. Block, Lunds Universitets Årsskrift, N. F. Afd. 2, 7, no. 1, (1911); G. C. Evans, American Mathematical Monthly, vol. 34, pp. 148–150.

‡ By definition $f(\epsilon) = O(\epsilon^\alpha)$ if $f(\epsilon)/\epsilon^\alpha$ is bounded.

bounded in the region $a \leqq x \leqq b$, $0 \leqq \epsilon \leqq \epsilon_0$. The integral equation

$$(1) \qquad u(x,\rho,\epsilon) + \lambda \int_a^b k(x,s,\rho,\epsilon)u(s,\rho,\epsilon)ds = f(x,\rho,\epsilon)$$

in which

$$k(x,s,\rho,\epsilon) \equiv k(x,s) + \rho h(x,s,\epsilon),$$
$$f(x,\rho,\epsilon) \equiv f(x) + \rho h(x,\epsilon),$$

reduces to (A_1) when $\rho = 0$ and to (A_2) when $\rho = \epsilon$. For any finite value of ρ, the Fredholm determinant and the Fredholm minors of $k(x, s, \rho, \epsilon)$ exist and are analytic for all finite values of λ. Also since $k(x, s, \rho, \epsilon)$ is analytic in ρ and possesses a resolvent for the particular value $\rho = 0$, the Fredholm determinant $D(\lambda, \rho, \epsilon)$ and the Fredholm minors

$$D\begin{pmatrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{pmatrix}, \lambda, \rho, \epsilon\Big), \qquad (\mu = 1,2,3, \cdots),$$

are analytic for all finite values of ρ, and are not identically zero.*

Let $\lambda = \lambda_0$ be a characteristic number of multiplicity p and index q† associated with the kernel $k(x, s, 0, \epsilon) \equiv k(x, s)$. Then‡

$$(2) \qquad D(\lambda,0,\epsilon) \equiv D(\lambda) = (\lambda - \lambda_0)^p d(\lambda), \qquad d(\lambda_0) \neq 0;$$

and

$$(3) \qquad D\begin{pmatrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{pmatrix}, \lambda, 0, \epsilon\Big) \equiv D\begin{pmatrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{pmatrix}, \lambda\Big)$$
$$= (\lambda - \lambda_0)^{q-\mu} d_\mu \begin{pmatrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{pmatrix}, \lambda\Big), \quad (\mu = 1,2, \cdots, q),$$

where, in general,

$$d_\mu \begin{pmatrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{pmatrix}, \lambda_0\Big) \neq 0.$$

* See, e.g. J. D. Tamarkin, Annals of Mathematics, (2), vol. 28 (1927), pp. 127–152.

† A characteristic number is said to be of index q if there are exactly q characteristic functions associated with it.

‡ These formulas follow as a direct consequence of the assumption that λ_0 is a characteristic number of multiplicity p and index q.

Also, for all values of μ, including $\mu=0$,*

$$\partial D\begin{pmatrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{pmatrix},\lambda,\rho,\epsilon\Big)/\partial\rho$$

$$= \sum_{i,j=1}^{\mu} (-1)^{i+j} h(x_i,s_j,\epsilon) D\begin{pmatrix} x_1 \cdots x_{i-1}x_{i+1} \cdots x_\mu \\ s_1 \cdots s_{j-1}\, s_{j+1} \cdots s_\mu \end{pmatrix},\lambda,\rho,\epsilon\Big)$$

$$+ \lambda D\begin{pmatrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{pmatrix},\lambda,\rho,\epsilon\Big) \int_a^b h(\xi,\xi,\epsilon)d\xi$$

$$(4)\quad -\lambda \sum_{i=1}^{\mu} \int_a^b h(x_i,\xi,\epsilon) D\begin{pmatrix} x_1 \cdots x_{i-1}\xi x_{i+1} \cdots x_\mu \\ s_1 \cdots s_{i-1}s_i s_{i+1} \cdots s_\mu \end{pmatrix},\lambda,\rho,\epsilon\Big)d\xi$$

$$-\lambda \sum_{i=1}^{\mu} \int_a^b D\begin{pmatrix} x_1 \cdots x_{i-1}x_i x_{i+1} \cdots x_\mu \\ s_1 \cdots s_{i-1}\,\xi\, s_{i+1} \cdots s_\mu \end{pmatrix},\lambda,\rho,\epsilon\Big) h(\xi,s_i,\epsilon)d\xi$$

$$-\lambda^2 \int_a^b h(\xi,\eta,\epsilon) D\begin{pmatrix} x_1 \cdots x_{\mu\eta} \\ s_1 \cdots s_{\mu\xi} \end{pmatrix},\lambda,\rho,\epsilon\Big)d\xi d\eta.$$

Assume that there is no characteristic number of $k(x, s)$ other than $\lambda=\lambda_0$ within or on the boundary of the region $|\lambda-\lambda_0|\leqq\delta_1$. The analytic function $D(\lambda, \rho, \epsilon)$ is represented in the region $|\lambda-\lambda_0|\leqq\delta_1$; $0\leqq\rho\leqq\delta_2<1$ by the convergent power series

$$(5)\quad D(\lambda,\rho,\epsilon) = \sum_{\alpha,\beta=0}^{\infty} A_{\alpha,\beta}(\epsilon)(\lambda-\lambda_0)^\alpha \rho^\beta, \qquad |A_{\alpha,\beta}(\epsilon)| \leqq A,$$

where A is a finite constant.

It follows from (2), (3), and (4) that

$$A_{\alpha,0}=0,\ \alpha<p, \qquad A_{\alpha,\beta}=0,\ \alpha+\beta<q, \qquad A_{p,0}\neq 0,$$

and, in general,

$$A_{\alpha,\beta}\neq 0, \quad \alpha+\beta=q.$$

We shall assume for the present that $A_{q-1,1}\neq 0$ and, by collecting terms of the same degree, express (5) in the form

* G. C. Evans, loc. cit.

$$D(\lambda,\rho,\epsilon) - D(\lambda) =$$
$$[A_{q-1,1}(\lambda-\lambda_0)^{q-1}\rho + A_{q-2,2}(\lambda-\lambda_0)^{q-2}\rho^2 + \cdots + A_{0,q}\rho^q]$$
$$+ [A_{q,1}(\lambda-\lambda_0)^q\rho + A_{q-1,2}(\lambda-\lambda_0)^{q-1}\rho^2 + \cdots + A_{0,q+1}\rho^{q+1}]$$
$$+ \cdots .$$

Consequently, for $\lambda-\lambda_0$ and ϵ sufficiently small,*

$$|D(\lambda,\epsilon) - D(\lambda)| < 2A\epsilon[|\lambda-\lambda_0| + \epsilon]^{q-1}.$$

Hence

$$D(\lambda,\epsilon) = D(\lambda)[1 + \psi(\lambda,\epsilon)]$$

where

$$|\psi(\lambda,\epsilon)| < 2A\epsilon[|\lambda-\lambda_0| + \epsilon]^{q-1}|D(\lambda)|^{-1}.$$

Consider a circle (γ) about λ_0 of radius $r \leqq \delta_1$. Since $d(\lambda)$ in (2) can not vanish on (γ), then on (γ) $|d(\lambda)| \geqq d_0^{-1} > 0$ where d_0 is a constant different from zero. If now we choose r so that

$$2Ad_0\epsilon[r+\epsilon]^{q-1}r^{-p} = C < 1 \tag{6}$$

then on (γ), $|\psi(\lambda,\epsilon)| < 1$ and the equation $D(\lambda,\epsilon)=0$ will have the same number of zeros within (γ) as $D(\lambda)=0$. It is readily seen that (6) implies $r = O(\epsilon^{1/(p-q+1)})$.

THEOREM 1. *If $\lambda=\lambda_0$ is a characteristic number of (A_1) of multiplicity p and index q, then for a fixed ϵ sufficiently small (A_2) will possess exactly p characteristic numbers $\lambda=\lambda_{0,i}(\epsilon)$ $(i=1, 2, \cdots, p)$ such that $\lambda_0-\lambda_{0,i}(\epsilon)=O(\epsilon^{1/(p-q+1)})$ at least.*

By an analogous method it is possible to discuss the special cases that arise when $A_{q-1,1}=0$. We shall find that if any of the coefficients $A_{\alpha,\beta}$, $\alpha+\beta=q$, are different from zero, then $r=O(\epsilon^{1/(p-q+1)})$; if all of the coefficients $A_{\alpha,\beta}, \alpha+\beta=q$, vanish, but one of the coefficients $A_{\alpha,\beta}$, $\alpha+\beta=q+1$, is different from zero, then $r=O(\epsilon^{1/(p-q)})$; if $A_{\alpha,\beta}$, $\alpha+\beta<q+2$, vanish, but one of the coefficients $A_{\alpha,\beta}$, $\alpha+\beta=q+2$ is different from zero, then $r=O(\epsilon^{1/(p+q-1)})$; and so forth.

* $D(\lambda, \epsilon, \epsilon)\equiv D(\lambda, \epsilon)$.

THEOREM 2. *If* $\{\phi_j(x)\}(j=1, 2, \cdots, q)$ *is a normalized set of characteristic functions of* (A_1) *associated with the characteristic number* $\lambda=\lambda_0$ *and* $\phi(x, \epsilon)$ *is a normalized characteristic function of* (A_2) *associated with the characteristic number* $\lambda=\lambda_{0,i}(\epsilon)$, *then there exists an approximately normalized* linear combination* $\sum_{j=1}^{q} c_j(\epsilon)\phi_j(x)$ *such that*

$$\phi(x,\epsilon) - \sum_{j=1}^{q} c_j(\epsilon)\phi_j(x) = O(\epsilon^{1/(p-q+1)})$$

at least.

Consider the equation

$$\phi(x,\epsilon)+\lambda_{0,i}(\epsilon)\int_a^b k(x,s,\epsilon)\phi(s,\epsilon)ds = 0, \tag{7}$$

where

$$\int_a^b |\phi(x,\epsilon)|^2 dx = 1.$$

This may be written as

$$\phi(x,\epsilon) + \lambda_0 \int_a^b k(x,s)\phi(s,\epsilon)ds = \Omega(x,\epsilon), \tag{8}$$

where

$$\Omega(x,\epsilon) = \lambda_0 \int_a^b [k(x,s) - k(x,s,\epsilon)]\phi(s,\epsilon)ds + [\lambda_0 - \lambda_{0,i}(\epsilon)] \int_a^b k(x,s,\epsilon)\phi(s,\epsilon)ds.$$

Since

$$\lambda_{0,i}(\epsilon) - \lambda_0 = O(\epsilon^{1/(p-q+1)}), \quad k(x,s,\epsilon) - k(x,s) = O(\epsilon),$$

and since $k(x, s, \epsilon)$ is bounded and $\phi(x, \epsilon)$ is normalized, we have $\Omega(x,\epsilon) = O(\epsilon^{1/(p-q+1)})$ at least.

But (8) is a non-homogeneous equation in which λ_0 is a characteristic number and $\phi(x, \epsilon)$ is a solution. Accordingly, if $R(x, s)$ is the pseudo-resolvent of $k(x, s)$, then

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* A function $\omega(x, \epsilon)$ is said to be approximately normalized if $\int_a^b |\omega(x, \epsilon)|^2 dx = 1+\eta(\epsilon)$ where $\eta(\epsilon)\to 0$ as $\epsilon\to 0$.

$$\phi(x,\epsilon) = \Omega(x,\epsilon) - \lambda_0 \int_a^b R(x,s)\Omega(s,\epsilon)ds + \sum_{j=1}^{q} c_j(\epsilon)\phi_j(x).$$

Since $R(x, s)$ is bounded,

$$(9) \quad \phi(x,\epsilon) = \sum_{j=1}^{q} c_j(\epsilon)\phi_j(x) + \eta(x,\epsilon), \quad \eta(x,\epsilon) = O(\epsilon^{1/(p-q+1)}),$$

which yields the result

$$\int_a^b \left| \sum_{j=1}^{q} c_j(\epsilon)\phi_j(x) \right|^2 dx = 1 + \bar{\eta}(x,\epsilon), \quad \bar{\eta}(x,\epsilon) = O(\epsilon^{1/(p-q+1)}).$$

THEOREM 3. *If $u(x)$ represents a solution of (A_1) corresponding to a fixed value of the parameter which is not a characteristic number, then for ϵ sufficiently small and the same value of the parameter there is a solution $u(x, \epsilon)$ of (A_2) such that*

$$u(x,\epsilon) - u(x) = O(\epsilon).$$

If λ is not a characteristic value of (A_1), then from Theorem 1 it follows that for ϵ sufficiently small λ is not a characteristic value of (A_2). We may write (A_2) as

$$u(x,\epsilon) + \lambda \int_a^b k(x,s)u(s,\epsilon)ds = g(x,\epsilon),$$

$$(x,\epsilon) = f(x,\epsilon) - \lambda \int_a^b [k(x,s,\epsilon) - k(x,s)]u(s,\epsilon)ds.$$

Let $\Gamma(x, s, \lambda)$ represent the resolvent of $k(x, s)$. Then

$$u(x,\epsilon) + \lambda \int_a^b \Gamma(x,s,\lambda)g(s,\epsilon)ds = g(x,\epsilon),$$

or

$$u(x,\epsilon) + \lambda \int_a^b \Gamma(x,s,\lambda,\epsilon)u(s,\epsilon)ds = g(x,\lambda,\epsilon)$$

where

$$\Gamma(x,s,\lambda,\epsilon) = k(x,s,\epsilon) - k(x,s) - \lambda \int_a^b \Gamma(x,\xi,\lambda)[k(\xi,s,\epsilon) - k(\xi,s)]d\xi,$$

$$g(x,\lambda,\epsilon) = f(x,\epsilon) - \lambda \int_a^b \Gamma(x,s,\lambda)f(s,\epsilon)ds.$$

Since $\mathfrak{k}(x, s, \lambda, \epsilon) = O(\epsilon)$, the Neumann series representing the resolvent of $\mathfrak{k}(x, s, \lambda, \epsilon)$ will converge and for ϵ sufficiently small will also be $O(\epsilon)$. Consequently, $u(x, \epsilon)$ is bounded.

If we set $v(x,\epsilon) = u(x, \epsilon) - u(x)$, then from (A_1) and (A_2) it follows that

$$v(x,\epsilon) + \lambda \int_a^b k(x,s)v(s,\epsilon)ds = \eta(x,\lambda,\epsilon),$$

where

$$\eta(x,\lambda,\epsilon) = f(x,\epsilon) - f(x) - \lambda \int_a^b [k(x,s,\epsilon) - k(x,s)]u(s,\epsilon)ds = O(\epsilon).$$

Hence

$$v(x,\epsilon) = \eta(x,\lambda,\epsilon) - \lambda \int_a^b \mathfrak{k}(x,s,\lambda)\eta(s,\lambda,\epsilon)ds = O(\epsilon),$$

which proves Theorem 3.

3. *Reduction of the Differential System and the Associated Difference System to Fredholm Integral Equations.* We consider the differential system represented in matrix notation*

$$(B_1) \qquad \begin{aligned} &L(Y\cdot) + \lambda Q(x)Y\cdot = F(x)\cdot, \\ &U(Y\cdot) \equiv W_0Y(0)\cdot + W_1Y(1)\cdot = 0, \ (0 \leqq x \leqq 1), \\ &L(Y\cdot) \equiv dY\cdot/dx - A(x)Y\cdot, \end{aligned}$$

where $A(x)$, $Q(x)$, and $F(x)\cdot$ are matrices whose elements satisfy a Lipschitz condition; W_0 and W_1 are matrices of constants; and λ is a scalar parameter.

Let the interval (0, 1) be divided into m equal parts by the points $x_0 \equiv 0, x_1, x_2, \cdots, x_m \equiv 1$. Then associated with the system (B_1) is a difference system

* Throughout the remainder of this paper capital letters—with the single exception of O—will be used to represent matrices with n rows and n columns. A dot following a capital indicates that the columns of the matrix are identical.

$$L(Y_l\cdot) + \lambda Q(x_l)Y_l\cdot = F(x_l)\cdot,$$

$$(B_2) \quad U(Y_l\cdot) \equiv W_0Y_0\cdot + W_1Y_m\cdot = 0, \quad (l = 1,2,\cdots,m),$$

$$L(Y_l\cdot) \equiv \Delta Y_{l-1}\cdot/\Delta x - A(x_l)Y_l\cdot\ ;\ \ \Delta Y_l\cdot = Y_{l+1}\cdot - Y_l\cdot\ ,$$

$$\Delta x = x_{l+1} - x_l = 1/m.$$

We may assume without loss of generality that $\lambda = 0$ is not a characteristic number of either (B_1) or (B_2). Then for an arbitrary $F(x)\cdot$ there exists a unique solution of the system

$$(10) \qquad \begin{aligned} L(Y\cdot) &= F(x)\cdot, \\ U(Y\cdot) &= 0, \end{aligned} \qquad (0 \leqq x \leqq 1),$$

and this solution can be expressed in the form

$$Y(x)\cdot = \int_0^1 G(x,s)F(s)\cdot ds,$$

where $G(x, s)$ is called the Green's matrix of the system and is given explicitly by the formula*

$$G(x,s) = \overline{G}(x,s) - Y_h(x)\left[U(Y_h(x)\right]^{-1}U\left[\overline{G}(x,s)\right],$$

where

$$\overline{G}(x,s) = \begin{cases} \frac{1}{2}Y_h(x)\left[Y_h(s)\right]^{-1}, & s < x \\ -\frac{1}{2}Y_h(x)\left[Y_h(s)\right]^{-1}, & s > x \end{cases}\ ;\ \ L(Y_h(x)) = 0.$$

Hence

$$(11) \qquad Y(x)\cdot + \lambda\int_0^1 K(x,s)Y(s)\cdot ds = \overline{F}(x)\cdot,$$

where

$$K(x,s) = G(x,s)Q(s), \qquad \overline{F}(x)\cdot = \int_0^1 G(x,s)F(s)\cdot ds,$$

is equivalent to (B_1). In the ordinary notation, the system (11) represents a system of n linear integral equations

* Birkhoff and Langer, Proceedings American Academy of Arts and Sciences, vol. 58 (1923), pp. 51–128.

$$y_i(x) + \lambda \int_0^1 \sum_{j=1}^{n} k_{i,j}(x,s) y_j(s) ds = \bar{f}_i(x),$$

(12)

$$k_{i,j}(x,s) = \sum_{\alpha=1}^{n} g_{i,\alpha}(x,s) q_{\alpha,j}(s), \quad (i,j = 1,2, \cdots, n).$$

If we define

$$k(x,s) = k_{i,j}(x - i + 1,\ s - j + 1), \quad 0 < \frac{x - i + 1}{s - j + 1} < 1,$$

$$f(x) = \bar{f}_i(x - i + 1), \quad 0 < x - i + 1 < 1,$$

then

$$y(x) + \lambda \int_0^n k(x,s) y(s) ds = f(x), \tag{13}$$

where $y(x) = y_i(x-i+1)$, $0<x-i+1<1$, is equivalent to (12).*

It is also possible by means of a Green's matrix to reduce the system (B_2) to a single Fredholm integral equation. For this purpose we shall need the following lemmas, the proofs of which do not present any difficulty:

i. *If* $Y_k^{(m)}(x_l)$, $(l=0, 1, 2, \cdots, m)$, *is a solution of the system*

$$L(Y_l) = 0, \qquad (l = 1,2, \cdots, m), \tag{14}$$

and if† $|Y_k^{(m)}(1)| \neq 0$, *there exists a number* m_0 *such that for all* $m > m_0$, $|Y_k^{(m)}(x_l)| \neq 0$, $(l=0, 1, 2, \cdots, m)$.

ii. *If* $|Y_k^{(m)}(1)| \neq 0$, *then for a sufficiently large m the most general solution of* (14) *is* $Y_k^{(m)}(x_l)C$, $(l=0, 1, 2, \cdots, m)$, *where C is a matrix of constants.*

iii. *If* $\overline{Y}^{(m)}(x_l)$, $(l=0, 1, 2, \cdots, m)$, *is a particular solution of the system*

$$L(Y_l) = F(x_l), \qquad (l = 1,2, \cdots, m), \tag{15}$$

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* I. Fredholm, Acta Matematica, vol. 27 (1903), pp. 378–379.

† The notation $|A|$ is used to indicate the determinant of the matrix A.

and if $|Y_k^{(m)}(1)| \neq 0$, *then for m sufficiently large*

$$Y^{(m)}(x_l) = \overline{Y}^{(m)}(x_l) + Y_k^m(x_l)C, \qquad (l = 0,1,2, \cdots, m),$$

is the most general solution of (15).

iv. *A particular solution of* (15) *is*

$$\overline{Y}^{(m)}(x_l) = \sum_{k=1}^{m} \overline{G}^{(m)}(x_l, x_{k-1})F(x_k)\Delta x, \qquad (l = 0,1,2, \cdots, m),$$

where

$$\overline{G}^{(m)}(x_l, x_{k-1}) = \begin{cases} \frac{1}{2}Y_k^{(m)}(x_l)\left[Y_k^{(m)}(x_{k-1})\right]^{-1}, & k \leqq l, \\ -\frac{1}{2}Y_k^{(m)}(x_l)\left[Y_k^{(m)}(x_{k-1})\right]^{-1}, & k > l. \end{cases}$$

By an argument closely analogous to that used by Birkhoff and Langer* for the differential system, it is readily proved that there exists a unique solution of the system

$$(16) \qquad L(Y_l \cdot) = F(x_l)\cdot, \quad U(Y_l \cdot) = 0, \quad (l = 1,2, \cdots, m),$$

and that this solution can be expressed in the form

$$Y^{(m)}(x_l)\cdot = \sum_{k=1}^{m} G^{(m)}(x_l, x_{k-1})F(x_k)\cdot\Delta x, \quad (l = 0,1,2, \cdots, m),$$

where $G^{(m)}(x_l, x_{k-1})$ is called the Green's matrix of the system and is given explicitly by the formula

$$G^{(m)}(x_l, x_{k-1}) = \overline{G}^{(m)}(x_l, x_{k-1}) - Y_k^{(m)}(x_l)\left\{U\left[Y_k^{(m)}(x_l)\right]\right\}^{-1}U\left[\overline{G}^{(m)}(x_l, x_{k-1})\right], \qquad (l = 0,1,2, \cdots, m;\ k = 1,2, \cdots, m).$$

Consequently the system

$$(17) \quad Y^{(m)}(x_l)\cdot + \lambda\sum_{k=1}^{m} G^{(m)}(x_l, x_{k-1})Q(x_k)Y^{(m)}(x_k)\cdot\Delta x = \overline{F}^m(x_l)\cdot,$$

where

$$\overline{F}^{(m)}(x_l)\cdot = \sum_{k=1}^{m} G^{(m)}(x_l, x_{k-1})F(x_k)\cdot\Delta x,$$

is equivalent to (B_2).

* Birkhoff and Langer, loc. cit.

On setting $G^{(m)}(x_l, x_{k-1}) = G^{(m)}(x,s)$ in the intervals

$$I_x: \quad x_{l-1} < x < x_l, \qquad (l = 1,2, \cdots, m),$$
$$I_s: \quad x_{k-1} < s < x_k, \qquad (k = 1,2, \cdots, m),$$

and

$$\overline{F}^{(m)}(x_l)\cdot = \overline{F}^{(m)}(x)\cdot, \quad Q(x_l) = Q^{(m)}(x), \quad Y^{(m)}(x_l)\cdot = Y^{m}(x)\cdot,$$

in the interval I_x, we reduce (17) to an equivalent system of integral equations

$$(18) \qquad Y^{(m)}(x)\cdot + \lambda \int_0^1 K^{(m)}(x,s)Y^{(m)}(s)\cdot ds = \overline{F}^{(m)}(x)\cdot,$$
$$K^{(m)}(x,s) = G^{(m)}(x,s)Q^{(m)}(s).$$

Then (18) is equivalent to

$$(19) \qquad y^{(m)}(x) + \lambda \int_0^n k^{(m)}(x,s)y^{(m)}(s)ds = f^{(m)}(x),$$

where

$$k^{(m)}(x,s) = k_{i,j}(x - i + 1, s - j + 1), \quad 0 < \frac{x - i + 1}{s - j + 1} < 1,$$
$$f^{(m)}(x) = \bar{f}_i^{(m)}(x - i + 1),$$
$$y^{(m)}(x) = y_i^{(m)}(x - i + 1), \qquad 0 < x - i + 1 < 1.$$

As a final step it is necessary to estimate the difference between the kernels $k(x, s)$ and $k^{(m)}(x, s)$. For this purpose we shall need the additional lemmas:

v. *If $U_1(x)$ and $U_2(x)$ are continuous matrices which, except at a finite number of points, satisfy the relations*

$$dU_1/dx = A(x)U_1 + \Omega_1, \quad U_1(1) = C_0,$$
$$dU_2/dx = A(x)U_2 + \Omega_2, \quad U_2(1) = C_0,$$

where $A(x) \ll a_0$; $\Omega_1 \ll \epsilon_1$ and $\Omega_2 \ll \epsilon_2$, ϵ_1 and ϵ_2 being positive scalars; then*

* $A << B$ indicates that each element of the matrix A is less in absolute value than the corresponding element of the matrix B. Similarly, $A << b$ indicates that each element of the matrix A is less in absolute value than the scalar b.

$$U_1(x) - U_2(x) \ll (\epsilon_1 + \epsilon_2)(e^{a_0 n x} - 1)/a_0 n.^*$$

vi. *The elements of the matrices*

$$Y_k(x), \quad [Y_k(x)]^{-1}, \quad U[Y_k(x)], \quad \{U[Y_k(x)\}^{-1}$$

are bounded; the same is true of the elements of the matrices

$$Y_k^{(m)}(x_l), \quad [Y_k^{(m)}(x_l)]^{-1}, \quad U[Y_k^{(m)}(x_l)], \quad \{U[Y_k^{(m)}(x_l)]\}^{-1},$$

$$U[\overline{G}^{(m)}(x_l, x_{k-1})], \quad (l = 0,1,2, \cdots, m\ ;\ k = 1,2, \cdots, m),$$

for m sufficiently large.†

vii. $$Y_k^{(m)}(x_l) - Y_k(x) = O(1/m) \quad \textit{in} \quad I_x.$$‡

Let the continuous matrix $Y_k^{(m)}(x)$ be defined in I_x by the relation

$$dY_k^{(m)}(x)/dx = \Delta Y_k^{(m)}(x_{l-1})/\Delta x = A(x_l)Y_l,$$

where $Y_l = Y_k^{(m)}(x_l)$. Then

$$dY_k^{(m)}(x)/dx = A(x)Y_k^{(m)}(x) + \Omega^{(m)}(x),$$

where

$$\Omega^{(m)}(x) = A(x_l)[Y_k^{(m)}(x_l) - Y_k^{(m)}(x)] + [A(x_l) - A(x)]Y_k^{(m)}(x).$$

It is evident that

$$\Omega^{(m)}(x) = O(1/m).$$

The lemma now follows as a consequence of (v).

viii. $$\overline{G}^{(m)}(x_l, x_{k-1}) - \overline{G}(x,s) = O(1/m) \quad \textit{in} \quad I_x, I_s.$$

ix. $$U[Y_k^{(m)}(x_l)] - U[Y_k(x)] = O(1/m).$$

x. $$\{U[Y_k^{(m)}(x_l)]\}^{-1} - \{U[Y_k(x)]\}^{-1} = O(1/m).$$

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* The method of proof is similar to that used in the case of a single differential equation. (Ch. J. de la Vallée Poussin, *Cours d'Analyse Infinitésimale*, vol. 2, ed. 4, p. 135.)

† See Birkhoff and Langer, loc. cit.

‡ $A(\epsilon) = O(\epsilon)$ if each element of $A(\epsilon)$ is $O(\epsilon)$.

Combining lemmas (v) to (x) and using the fact that $Q(x)$ and $F(x)\cdot$ satisfy a Lipschitz condition, we obtain the final lemmas:

xi. $G^{(m)}(x_l, x_{k-1}) - G(x,s) = O(1/m)$ *in* I_x, I_s.

xii. $$K^{(m)}(x,s) - K(x,s) = O(1/m)\ ;$$

$$k^{(m)}(x,s) - k(x,s) = O(1/m)\ ;$$

$$f^{(m)}(x) - f(x) = O(1/m).$$

Equations (13) and (19) can now be treated by methods developed in §2.

4. *Reduction of the Integro-Differential System and the Associated Difference System to Fredholm Integral Equations.* Let us consider the integro-differential system

$$(C_1)\qquad \begin{aligned} L(Y\cdot) + \lambda Q(x)Y\cdot &= F(x)\cdot + \int_0^1 H(x,s)Y(s)\cdot ds, \\ U(Y\cdot) &= 0, \end{aligned}$$

where $L(Y\cdot)$, $U(Y\cdot)$, $Q(x)$, $F(x)\cdot$, and λ have been defined above, and $H(x, s)$ satisfies a Lipschitz condition in (x, s).

Associated with this system is a difference system

$$(C_2)\qquad \begin{aligned} L(Y_l\cdot) + \lambda Q(x_l)Y_l\cdot &= F(x_l)\cdot + \sum_{k=0}^{m} H(x_l, x_k)Y_k\cdot\Delta x, \\ U(Y_l\cdot) &= 0, \qquad (l = 1,2,\cdots, m). \end{aligned}$$

We can assume without loss of generality that $\lambda = 0$ is not a characteristic number* of either (C_1) or the differential system

* The discussion of characteristic numbers and characteristic functions of a single integro-differential equation satisfying boundary conditions, has been given by J. D. Tamarkin (Transactions of this Society, vol. 29 (1927), pp. 775–800). His results have been extended to a system of integro-differential equations by F. C. Jonah in a thesis to be submitted to Brown University in candidacy for the degree of Doctor of Philosophy.

(B_1). Since $\lambda = 0$ is not a characteristic number of (C_1), the system

$$(20)\qquad \begin{aligned} L(Y\cdot) &= F(x)\cdot + \int_0^1 H(x,s)Y(s)\cdot ds, \\ U(Y\cdot) &= 0, \end{aligned}$$

possesses a unique solution for an arbitrary $F(x)\cdot$ and this can be expressed in the form $Y(x)\cdot = \int_0^1 \Gamma(x, s)F(s)\cdot ds$. The matrix $\Gamma(x, s)$ is uniquely determined at its points of continuity and is called the Green's matrix of (20). It is readily seen that $\Gamma(x, s)$ is given explicitly by the formula

$$\Gamma(x,s) = G(x,s) + \int_0^1 \mathfrak{H}(x,\xi)G(\xi,s)d\xi,$$

if $\mathfrak{H}(x, s)$ is the resolvent of the matrix $\overline{H}(x, s)$, and

$$Y(x)\cdot = \int_0^1 \overline{H}(x,s)Y(s)\cdot ds + \overline{F}(x)\cdot,$$

$$(21)\qquad \overline{H}(x,s) = \int_0^1 G(x,\xi)H(\xi,s)d\xi,$$

$$\overline{F}(x)\cdot = \int_0^1 G(x,s)F(s)\cdot ds.$$

Hence

$$(22)\quad Y(x)\cdot + \lambda\int_0^1 \Gamma(x,s)Q(s)Y(s)\cdot ds = \int_0^1 \Gamma(x,s)F(s)\cdot ds$$

is equivalent to (C_1).

Similarly, an explicit expression for the Green's matrix of the system

$$(23)\qquad \begin{aligned} L(Y_l\cdot) &= F(x_l)\cdot + \sum_{k=0}^{m} H(x_l,x_k)F(x_k)\cdot\Delta x, \\ U(Y_l\cdot) &= 0, \qquad\qquad (l = 1,2,\cdots,m), \end{aligned}$$

can be obtained. Since $\lambda = 0$ is not a characteristic number of

(B_1), then from Theorem 1 it follows that for m sufficiently large $\lambda=0$ is not a characteristic number of (B_2). Hence

$$Y^{(m)}(x_l)\cdot = \sum_{k=0}^{m}\overline{H}^{(m)}(x_l,x_k)Y^{(m)}(x_k)\cdot\Delta x+\overline{F}(x_l)\cdot, \qquad (l=0,1,2,\cdots,m),$$

$$\overline{H}^{(m)}(x_l,x_k) = \sum_{j=1}^{m}G^{(m)}(x_l,x_{j-1})H(x_j,x_k)\Delta x,$$

$$\overline{F}(x_l)\cdot = \sum_{k=1}^{m}G^{(m)}(x_l,x_{k-1})F(x_k)\cdot\Delta x,$$

or

$$Y^{(m)}(x)\cdot = \int_0^1\overline{H}^{(m)}(x,s)Y^{(m)}(s)ds+\overline{F}^{(m)}(x)\cdot, \tag{24}$$

where

$$H^{(m)}(x,s)=H(x_l,x_k) \quad \text{in} \quad I_x,I_s,$$

$$\overline{H}^{(m)}(x,s)=\int_0^1 G^{(m)}(x,\xi)H^{(m)}(\xi,s)d\xi=\overline{H}^{(m)}(x_l,x_k) \ \textit{in}\ I_x,I_s,$$

$$\overline{F}^{(m)}(x)=\int_0^1 G^{(m)}(x,s)F^{(m)}(s)\cdot ds=\overline{F}(x_l)\cdot \ \textit{in}\ I_x,$$

is equivalent to (23).

In virtue of lemma (xi) and the assumption that $H(x, s)$ satisfies a Lipschitz condition, we have the lemma

xiii. $$\overline{H}^{(m)}(x,s)-H(x,s)=O(1/m),$$
$$\overline{F}^{(m)}(x)\cdot-\overline{F}(x)\cdot=O(1/m).$$

In consequence of Theorem 1, since (21) has a unique solution, then for m sufficiently large (24) will also possess a unique solution. So, for m sufficiently large, there exists a resolvent matrix $\Gamma^{(m)}(x, s)$ associated with $\overline{H}^{(m)}(x, s)$, and

$$Y^{(m)}(x)\cdot = \int_0^1\Gamma^{(m)}(x,s)F^{(m)}(s)\cdot ds,$$

where

$$\Gamma^{(m)}(x,s) = G^{(m)}(x,s) + \int_0^1 \mathfrak{H}^{(m)}(x,\xi)G^{(m)}(\xi,s)d\xi.$$

Hence

$$Y^{(m)}(x) \cdot + \lambda \int_0^1 \Gamma^{(m)}(x,s)Q^{(m)}(s)Y^{(m)}(s) \cdot ds = \int_0^1 \Gamma^{(m)}(x,s)F^{(m)}(s) \cdot ds, \tag{25}$$

is equivalent to (C_2), for $l<m$.

It is necessary to estimate the order of difference between the Green's matrices $\Gamma^{(m)}(x, s)$ and $\Gamma(x, s)$. For this purpose we need the lemma

xiv. $$\mathfrak{H}^{(m)}(x,s) - \mathfrak{H}(x,s) = O(1/m).$$

The resolvent matrices $\mathfrak{H}^{(m)}(x, s)$, $\mathfrak{H}(x, s)$ and the kernels $\overline{H}^{(m)}(x, s)$, $\overline{H}(x, s)$ satisfy the relations

$$\mathfrak{H}(x,s) - \overline{H}(x,s) = \int_0^1 \overline{H}(x,\xi)\mathfrak{H}(\xi,s)d\xi = \int_0^1 \mathfrak{H}(x,\xi)\overline{H}(\xi,s)d\xi,$$

$$\mathfrak{H}^{(m)}(x,s) - \overline{H}^{(m)}(x,s) = \int_0^1 \overline{H}^{(m)}(x,\xi)\mathfrak{H}^{(m)}(\xi,s)d\xi = \int_0^1 \mathfrak{H}^{(m)}(x,\xi)\overline{H}^{(m)}(\xi,s)d\xi.$$

Let

$$\overline{H}^{(m)}(x,s) - \overline{H}(x,s) = H_1^{(m)}(x,s) = O(1/m),$$
$$\mathfrak{H}^{(m)}(x,s) - \mathfrak{H}(x,s) = \mathfrak{U}^{(m)}(x,s).$$

Then

$$\mathfrak{U}^{(m)}(x,s) = \int_0^1 \overline{H}^{(m)}(x,\xi)\mathfrak{U}^{(m)}(\xi,s)d\xi + \mathfrak{F}_1^{(m)}(x,s),$$

$$\mathfrak{F}_1^{(m)}(x,s) = H_1^{(m)}(x,s) + \int_0^1 H_1^{(m)}(x,\xi)\mathfrak{H}(\xi,s)d\xi = O(1/m). \tag{26}$$

The equation (26) is equivalent to

$$\mathfrak{U}^{(m)}(x,s) = \int_0^1 \overline{H}(x,\xi)\mathfrak{U}^{(m)}(\xi,s)d\xi + \mathfrak{F}_3^{(m)}(x,s)\,,$$

$$\mathfrak{F}_2^{(m)}(x,s) = \mathfrak{F}_1^{(m)}(x,s) + \int_0^1 H_1^{(m)}(x,\xi)\mathfrak{U}^{(m)}(\xi,s)d\xi.$$

But $\overline{H}(x, s)$ possesses a resolvent $\mathfrak{H}(x, s)$. Hence

$$\mathfrak{U}^{(m)}(x,s) = \int_0^1 H_2^{(m)}(x,\xi)\mathfrak{U}^{(m)}(\xi,s)d\xi + \mathfrak{F}_3^{(m)}(x,s),$$

where

$$H_2^{(m)}(x,s) = H_1^{(m)}(x,s) + \int_0^1 \mathfrak{H}(x,\xi)H_1^{(m)}(\xi,s)d\xi = O(1/m),$$

$$\mathfrak{F}_3^{(m)}(x,s) = \mathfrak{F}_1^{(m)}(x,s) + \int_0^1 \mathfrak{H}(x,\xi)\mathfrak{F}_1^{(m)}(\xi,s)d\xi = O(1/m).$$

But for m sufficiently large, the elements of $\mathfrak{H}_2^{(m)}(x, \xi)$, the resolvent matrix of $H_2^{(m)}(x, \xi)$, are bounded, and

$$\mathfrak{U}^{(m)}(x,s) = \int_0^1 \mathfrak{H}_2^{(m)}(x,\xi)\mathfrak{F}_3^{(m)}(\xi,s)d\xi + \mathfrak{F}_3^{(m)}(x,s),$$

whence

$$\mathfrak{U}^{(m)}(x,s) = O(1/m).$$

In virtue of this lemma,

$$\Gamma^{(m)}(x,s) - \Gamma(x,s) = O(1/m).$$

We have shown that the systems (C_1) and (C_2) can be reduced respectively to the integral equations (22) and (25), respectively which possess the same character as the integral equations (11) and (18). So, (22) and (25) can be reduced to Fredholm integral equations to which the theorems of §2 apply.

THE UNIVERSITY OF ROCHESTER

LINNAEUS WAYLAND DOWLING

IN MEMORIAM

The following minute has been adopted by the Faculty of the University of Wisconsin, and is printed here in memory of Professor Dowling, who had been a member of this Society since 1897.

Linnaeus Wayland Dowling was born at Medina, Michigan, on Decem ber 7, 1867. He graduated from Adrian College in 1890 and obtained the degree of Doctor of Philosophy at Clark University in 1895. He gave his entire collegiate service of thirty-three years to the University of Wisconsin, becoming successively Instructor, Assistant Professor, Associate Professor, and Professor of Mathematics. His special field was geometry, and he was the author of textbooks on Analytic Geometry, Projective Geometry, and Mathematics of Insurance.

Professor Dowling's instruction was characterized by an exceptional clarity of exposition which, with his magnetism, geniality of manner, and interest in his students, made him a universally popular and beloved teacher. He was a man of unusual breadth of interests: a talented musician, an eager and well-informed botanist, and a student of the humanities. His knowledge of the language and literature of Italy, where he studied under the geometer Segre, was exceptional; and he was an ardent reader and lover of poetry. These talents explain, in part, the bonds of sympathy that united him with so many colleagues and students. But above all his attraction was due to his personal qualities, his cheerful disposition and his kindly character. His influence in these private associations was no less great than in his admirable public service as a teacher, and he will long be remembered and missed by his friends.

Be it resolved that we as a faculty hereby express our sense of loss in the death of our colleague, Professor Dowling, and that this resolution be spread upon our minutes and a copy of it sent to Mrs. Dowling and to his son Dr. Philip Dowling.

[Resolution of the Faculty of the University of Wisconsin]

GRASSMANN'S PROJECTIVE GEOMETRY, VOLUME II

Projektive Geometrie der Ebene unter Benutzung der Punktrechnung. Volume II: *Ternäres.* Part 1. By Hermann Grassmann. Leipzig und Berlin, B. G. Teubner, 1913. xii+410 pp.

Projektive Geometrie der Ebene unter Benutzung der Punktrechnung. Volume II: *Ternäres.* Part 2. By Hermann Grassmann. Leipzig und Berlin. B. G. Teubner, 1927. xvi+522 pp.

The first volume of this work was published in 1909 and reviewed in this Bulletin (1913) by L. W. Dowling. The first part of the second volume appeared in 1913, but, owing to the world war, the volume was not completed before December 1921, when the finished second part was deposited as manuscript in the author's desk. Hermann Grassmann, Jr. died the following month, Jan. 21, 1922. One of his students, G. Wolff in Hannover, took charge of the publication of this second part, but owing to financial difficulties the work did not appear before 1927. Among those giving financial aid we note our own E. Carus.

The first part, which has four chapters (Hauptteile), introduces us to the ternary field of projective geometry. A point x in the plane is given by the equation $x=\xi_1 e_1+\xi_2 e_2+\xi_3 e_3$, where e_1, e_2, e_3 are the vertices of the fundamental triangle; ξ_1, ξ_2, ξ_3, the "Ableitszahlen," may be considered as the Grassmann coordinates of the point x. The dual of the point is the "stab," or directed line-segment which is bound to the line on which it lies, in the sense that it can only be carried along the line in its own direction or in the opposite direction. Any stab U in the plane is represented by the equation $U=u_1E_1+u_2E_2+u_3E_3$, where u_1, u_2, u_3 are the Grassmann line-coordinates and E_1, E_2, E_3 are "stäbe" on the lines joining the fundamental points, or "grundstäbe," defined by the equations $E_1=[e_2e_3]$, $E_2=[e_3e_1]$, $E_3=[e_1e_2]$. The unit point e, mass m, is now defined as a point with coordinates (1, 1, 1), so that $e=e_1+e_2+e_3$; the masses m_1, m_2, m_3 may be so chosen that the exterior three-point product $[e_1e_2e_3]=1$ (blatteinheit). Dually we have a unit "stab" of length l such that $E=E_1+E_2+E_3$ and $[E_1E_2E_3]=[e_1e_2e_3]=1$. It is then shown that the ratios $\xi_1:\xi_2:\xi_3$ and $u_1:u_2:u_3$ may be geometrically interpreted as double ratios

$$\frac{\xi_2}{\xi_3}=\frac{p_2}{p_2'}:\frac{p_3}{p_3'},\qquad \frac{\xi_3}{\xi_1}=\frac{p_3}{p_3'}:\frac{p_1}{p_1'},\qquad \frac{\xi_1}{\xi_2}=\frac{p_1}{p_1'}:\frac{p_2}{p_2'},$$

where p_1, p_2, p_3 are the distances of the point x from the sides of the fundamental triangle, and p_1', p_2', p_3' those of the unit point e from the same sides respectively. Dually we obtain for the ratios $u_1:u_2:u_3$ the values

$$\frac{u_2}{u_3}=\frac{q_2}{q_2'}:\frac{q_3}{q_3'},\qquad \frac{u_3}{u_1}=\frac{q_3}{q_3'}:\frac{q_1}{q_1'},\qquad \frac{u_1}{u_2}=\frac{q_1}{q_1'}:\frac{q_2}{q_2'},$$

q_1, q_2, q_3 being the lengths of the perpendiculars from the vertices e_1, e_2, e_3 on the stab U, and q_1', q_2', q_3' those of the perpendiculars from the same

three vertices on the unit stab E. By letting E_3 recede to infinity the author obtains the Hessian coordinate system familiar to students of Clebsch-Lindemann. The chapter concludes with a section on the harmonic relations in the quadrilateral-quadrangular configuration.

The Grassmann algorithm when applied to projective geometry does not however produce a pure coordinate geometry. The coordinates ξ_i and u_i are used in proving certain important formulas involving exterior and combinatory products, and the geometric and projective properties are then derived from the interpretation of these formulas; in other words, direct analysis is employed as far as possible. That coordinates cannot be entirely dispensed with seems to substantiate Study's dictum, at least in this case, that "without coordinates there is no geometry."

The second chapter deals with the fundamental properties of collineations, their double-elements and certain combinatory products of two and three collineations. A collineation k is defined by the symbolic or "extensive" fraction

$$k = \frac{a_1, a_2, a_3}{e_1, e_2, e_3},$$

which means that $e_1k = a_1$, $e_2k = a_2$, $e_3k = a_3$, and that $(\xi_1e_1 + \xi_2e_2 + \xi_3e_3)k = \xi_1a_1 + \xi_2a_2 + \xi_3a_3$, or $xk = y$; to the vertices e_1, e_2, e_3 of the fundamental triangle correspond the vertices a_1, a_2, a_3 of another triangle. Any point-row $y + hz$ is transformed into a point-row $yk + hkz = (\xi_1 + h\zeta_1)a_1 + (\xi_2 + h\zeta_2)a_2 + (\xi_3 + h\zeta_3)a_3$, and the double-ratio of four points x, y, z, u on a line is invariant.

Closely related to the collineation k is the so-called adjoint collineation

$$K = \frac{A_1, A_2, A_3}{E_1, E_2, E_3}$$

which is a line-to-line transformation (Stab-Stab Abbildung) carrying a stab U into another stab UK, that is $U = \sum u_iE_i$ is transformed into $V = \sum u_iA_i$, where $E_1 = [e_2e_3]$, $E_2 = [e_3e_1]$, $E_3 = [e_1e_2]$, $A_1 = [a_2a_3]$, $A_2 = [a_3a_1]$, $A_3 = [a_1a_2]$. From the equation $[x, U] = 0$, which means that x is on U, it follows that $[xk, UK] = 0$, that is, the transform xk of the point x by k is on the transform UK of U by K.

If k and l are two collineations, the combinatory product $[kl]$ is defined as follows:

$$(1) \qquad [kl] = \frac{[yz \cdot kl]}{[yz]} = \frac{[yk \cdot zl] - [zk \cdot yl]}{2[yz]},$$

the right side of which is shown not to depend on the two points y and z; hence the symbol $[kl]$ is justified. If then in (1) we put $y = e_1$, $z = e_2$; $y = e_2$, $z = e_3$; $y = e_3$, $z = e_1$ in succession, it is easily seen that $[kl]$ may be put in the form of an extensive fraction

$$[kl] = \frac{[e_2e_3 \cdot kl], [e_3e_1 \cdot kl], [e_1e_2 \cdot kl]}{[e_2e_3], [e_3e_1], [e_1e_2]}.$$

If $k = l$, we have $[k^2] = K$, an important formula.

The combinatory product of three collineations is defined thus:

$$(2)\quad [klm] = \frac{[xyz\cdot klm]}{[xyz]} = \frac{[xk\cdot yl\cdot zm] + [yk\cdot zl\cdot xm] + [zk\cdot xl\cdot ym] - [xk\cdot zl\cdot ym] - [yk\cdot xm\cdot zl] - [zk\cdot ym\cdot xl]}{6[xyz]},$$

and it is shown that the fraction on the right side does not depend on the points x, y, z, only on the numerators a_i, l_i, c_i of the extended fractions k, l and m. If $k=l=m$, we have the combinatory cube

$$[k^3] = \frac{[a_1a_2a_3]}{[e_1e_2e_3]} = \frac{[e_1k\cdot e_2k\cdot e_3k]}{[e_1e_2e_3]}$$

which is another fundamental formula. The author then proceeds to classify the various singular collineations, namely those for which $[k^3]=0$:

1. $[a_1a_2a_3]=0$, at least one of the products $[a_ia_k]$ being different from zero.
2. $[a_1a_2a_3]=0$, all the products $[a_ia_k]$ vanishing.
3. $[a_1a_2a_3]=0$, all three a's vanishing.

Starting with the adjoint K and the adjoint to K, denoted by $\bar{k}$,

$$\bar{k} = \frac{[A_2A_3], [A_3A_1], [A_1A_2]}{[E_2E_3], [E_3E_1], [E_1E_2]}$$

and introducing the combinatory product $[Kl]$ of two stab-stab collineations, he proves that $[K^2]=\bar{k}=ak$, where $a=[a_1a_2a_3]=[k^3]$. The combinatory product $[KLM]$ is then defined in a manner analogous to that for $[klm]$ and an expression for $[K^3]$ is obtained, viz.: $[K^3]=[A_1A_2A_3]=[a_1a_2a_3]=a$. A classification of singular stab-stab collineations then follows, dual to the above for point-collineations.

It appears here, as elsewhere throughout the work, that the author discusses the dual case in detail instead of merely stating the result, applying the principle of duality, as is usually done, in projective geometry. This tends to increase materially the number of calculations and becomes wearisome in the long run, but the elementary student who attacks the subject for the first time *via* Grassmann will perhaps be satisfied, as he is saved the trouble of proving the dual case. It is all "cut and dry." In the second part of the volume he sometimes, but not often, omits the calculation.

The double elements of the collineation are obtained starting with the condition $d_ik=r_id_i$ where d_i is a point and r_i a pure number. The question is then: What point, or points d_i are left in situ in a collineation k, the point d_i changing at most its mass which is multiplied by a number r_i? The answer to this question leads to a cubic equation having r_1, r_2, r_3 for roots. This is *the characteristic equation*. We are thus led to a collineation

$$k = \frac{r_1d_1, r_2d_2, r_3d_3}{d_1, d_2, d_3},$$

d_1, d_2, d_3 being the fundamental triangle. The dual case, which is carried out in detail, gives rise to the stab-stab collineation

$$K = \frac{r_2r_3[d_2d_3], r_3r_1[d_3d_1], r_1r_2[d_1d_2]}{[d_2d_3], [d_3d_1], [d_1d_2]}.$$

The various subcases are then considered, the result being a slightly modified form of the classification found in Veblen and Young's treatise, vol. I, pp. 271–276. It may be noted in this connection that Grassmann needs 25 pages to exhaust the subject of double elements and types of collineations, while Veblen and Young need only 5 pages. This is not due to a more complicated algorithm, for the calculations in Grassmann are simple, and the geometry can be read from the symbolic equations without difficulty, but rather to a more detailed consideration of the geometry involved. In this respect the method of Grassmann is carried out very carefully; he never forgets that he is a geometer.

The last section (29) of this chapter deals with the geometric meaning of the vanishing of the combinatory product $[klm]$ which was defined above, (eq. 2). If $l=m$ we get the product $[kll]=0$ or

$$[xkyz] + [ykzx] + [zkxy] = 0. \tag{3}$$

If (3) is satisfied the collineation has an *inscribed triangular position.* This theorem is due to Pasch. A few more theorems of a similar kind then follow. The remainder of the chapter deals with point-triples. The locus of points x such that the two pencils $xabc$ and $xa'b'c'$ are in involution is a cubic called the cubic of involution; the special case when this cubic degenerates into three lines, i.e. when a, b, c and a', b', c' are collinear point triples is then considered, and a number of theorems on point-triples are deduced.

In the sixth chapter the author considers first correlations in the plane and then the more special polarities with their pole-curves of the second order and class. He sets up the extensive fractions for correlations and and polarities together with a number of combinatory products of two and three correlations (polarities). Starting with these formulas, the remaining part of the chapter is devoted to the consideration of poles and polars, the polar triangle, curves (that is pole-curves) of the second order and class. etc. The discussion of singular or "entartete" systems of polarity occupies about 30 pages, and the various forms of conics, considered as point-loci and stab-loci and depending on the various positions of their polar triangles, another 57 pages. The last section deals with conics in Cartesian point-coordinates (oblique axes) and Hessian line-coordinates, one side of the polar triangle being at infinity. The last, but not the least interesting, chapter deals with the theory of pencils of conics and their duals, ranges of conics. We shall not give an account of the various cases considered. The work is done with meticulous care, and the dual case (ranges) is treated in detail. The author finds no place for a discussion of group-properties of collineations, correlations and polarities, although in the first volume he did consider the product (folgeprodukt) kl of two projectivities in the binary field. The notion of an invariant, as well as invariant configurations connected with the various subgroups of projective transformations, is also absent. Nevertheless, the first part of this volume is a decided success from the standpoint of the Grassmann point-calculus; he no doubt had two main objects in view: To write a projective geometry based on the Grassmann point-calculus, and to make it a magisterial work that a

student can easily master. Any mathematician who has found the elder Grassmann's presentation of his point-calculus, such as is found in Grassmann's *Ausdehnungslehre* of 1862 rather "abschreckend" will be pleasingly surprised at the ease with which the calculus can be mastered when presented in a masterly way.

The second part begins with a section on the projective geometry on a curve of the second order and class. Involutions on a conic are carefully considered and a following short section deals with the equianharmonic projectivities.

In the next chapter the author turns to the consideration of range-pencils and pencil-ranges of conics. Special cases are considered such as homo-asymptotic hyperbolas and ellipses. All collineations are then found that leave pencil-ranges invariant, and incidentally also the collineations which transform a conic into itself. The general theory of reciprocities (correlations) is treated in the remainder of the chapter. Let

$$r = \frac{A_1, A_2, A_3}{e_1, e_2, e_3}, \quad R = r^2 = \frac{a_1, a_2, a_3}{E_1, E_2, E_3}$$

be a reciprocity and its adjoint. If $[A_1A_2A_3] \neq 0$, and $[a_1a_2a_3] \neq 0$, we have also the inverse

$$\frac{1}{r} = \frac{e_1, e_2, e_3}{A_1, A_2, A_3}, \qquad \frac{1}{R} = \frac{E_1, E_2, E_3}{a_1, a_2, a_3};$$

for the special case of polarity we have $r = a/R$, $a = [a_1a_2a_3]$. The curves $[x \cdot xr] = 0$, and $[U \cdot UR] = 0$ are called the *nuclei* of the reciprocity. The first expresses the condition that a point shall lie on the corresponding line, and the second equation that the line U shall pass through the corresponding point. These curves are also called respectively *the pole-curve* and the *polar curve*. The reciprocity conjugate to r is

$$r' = \frac{A_1', A_2', A_3'}{e_1, e_2, e_3} = \frac{a}{R},$$

where $A_i' = a_{1i}E_1 + a_{2i}E_2 + a_{3i}E_3$. A null-system of the second order is one for which $a_{ii} = 0$, $a_{ik} = -a_{ki}$, and for a null-system of the second class we have $A_{ii} = 0$, $A_{ik} = -A_{ki}$, the A_{ik} being the minors of the determinant $|a_{11}a_{22}a_{33}|$. A reciprocity of this kind may be put in the form

$$n = \frac{[ae_1], [ae_2], [ae_3]}{e_1, e_2, e_3},$$

in which a stands for the null-point of the reciprocity. Dually we get

$$N = \frac{[AE_1], [AE_2], [AE_3]}{E_1, E_2, E_3},$$

where A is the null-axis of the reciprocity. Of these two singular correlations, the one carries a point x into a line nx joining x to the null-point a so that the pole-curve becomes a double point a, and the second carries a line U into the point of intersection of the line with the null-axis, i.e. A, as a double line, is the polar curve.

The author then discusses conjugate and adjoint reciprocities and the relative positions of their nuclei. Although for a reciprocity, which is not a polarity, we must always have $r \neq r'$, the equality $cr = cr'$ may be satisfied for a special point. This point is shown to have the coordinates $c_1 : c_2 : c_3 = a_{23} - a_{32} : a_{31} - a_{13} : a_{12} - a_{21}$ and is called the "kernpunkt" of the reciprocity, and dually, for the coordinates of the "kerngerade" $C_1 : C_2 : C_3 = A_{23} - A_{31} : A_{32} - A_{13} : A_{12} - A_{21}$, and he proves that the "kernpunkt" of a reciprocity is the pole of the "kerngerade" with respect to both nuclei of the reciprocity.

In the tenth and eleventh chapters the author takes up the subject of apolarity. He derives the condition for apolarity of two reciprocities in the form

$$[rS] = \frac{1}{3[xyz]}\{[xr \cdot XS] + [yr \cdot YS] + [zr \cdot ZS]\} = 0, \tag{4}$$

where

$$X = [yz], \quad y = [zx], \quad Z = [xy], \quad S = [st].$$

If in this equation we put $e_i r = A_i$ and $E_i S = b_i$, $x = e_1$, $y = e_2$, $z = e_3$, $X = E_1$, $Y = E_2$, $Z = E_3$, the condition $[rS] = 0$ takes the well known form

$$a_{11}B_{11} + \cdots + (a_{22}B_{22} + a_{33}B_{33}) + \cdots = 0,$$

and when r and S are two polar systems p and Q, $a_{11}B_{11} + \cdots + 2a_{23}B_{23} + \cdots = 0$.

In Chapter XI the author introduces two new symbolic expressions, the *Lückenform* and the *Potenzform*. We shall not try to give English equivalents for these terms; *lacunary forms* would perhaps do for the first term, but *powerform* does not sound well, and we shall therefore use the German terms throughout. Suppose we have a polarity of the second order

$$p = \frac{A_1, A_2, A_3}{e_1, e_2, e_3},$$

where $A_i = \sum a_{ik}E_k$ and $a_{ik} = a_{ki}$; then if $x = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3$, we have the polar $xp = \xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3$. Multiplying x exteriorly by E_1, E_2, E_3 we get $[xE_1] = \xi_1$, $[xE_2] = \xi_2$, $[xE_3] = \xi_3$ and the polar xp becomes

$$xp = [xE_1]A_1 + [xE_2]A_2 + [xE_3]A_3 = \sum [xE_i]A_i,$$

and the quadratic form $[x \cdot xp]$ is represented by the expression

$$[x \cdot xp] = [xE_1][xA_1] + [xE_2][xA_2] + [xE_3][xA_3] = \sum_1^3 [xE_i][xA_i].$$

Since x occurs twice in each term on the right side, we may, if we wish, put x^2 outside and fill in the spaces left vacant by a letter l (Lücke:lacuna) so that the quadratic form is written,

$$[x \cdot xp] = \{[lE_1][lA_1] + [lE_2][lA_2] + [lE_3][lA_3]\}x^2 = \sum_1^3 [lE_i][lA_i]x^2. \tag{5}$$

The factor of x^2 is called a "Lückenform" of the second order; the small letter l in the expression means that the space is to be filled in by a point

such as x, y, etc.; l is therefore so to speak "the ghost of a departed point." We denote the lückenform by the letter A_2 and write

$$A_2 = \sum_1^3 [lE_i][lA_i]$$

and, introducing for the A's their values $\sum a_{ik}E_k$, we have

$$A_2 = a_{11}[lE_1]^2 + a_{22}[lE_2]^2 + a_{33}[lE_3]^2 + 2a_{23}[lE_2][lE_3] + 2a_{31}[lE_1][lE_3] + 2a_{12}[lE_1][lE_2]$$

and $[x\cdot xp]=A_2x^2$, so that $A_2x^2=0$ becomes the equation of the polecurve. In the same way we have for the bilinear form $[y\cdot xp]=\sum[yE_i][xA_i]$; but, since for any polarity we have the fundamental equation $[y\cdot xp]=[x\cdot yp]$, we have

$$[x\cdot yp] = [y\cdot xp] = \sum_1^3 \frac{[xE_i][yA_i] + [yE_i][xA_i]}{2},$$

so that we may write

$$[x\cdot yp] = [y\cdot xp] = A_2xy = \sum[lE_i][lA_i]\{yx\} = \frac{[xE_i][yA_i] + [yE_i][xA_i]}{2} = A_2yx;$$

xy is an algebraic product since $A_2xy=A_2yx$.

It is important to note that each term of (5) is a product of two factors which may be considered as lacunary forms of the first order and the product is not changed by interchanging the factors; the product $[xE_1][xA_1]$, for example, vanishes whenever the point is on the line-pair containing the stäbe E_1 and A_1, and in the case of the quadratic terms the line-pair becomes a double line. We may therefore say that the products in the separate terms of the lückenform (5) represent line-pairs, considered as reducible curves of the second order. In order to simplify the expression we leave out the l's and the brackets; the remaining stab-factors $E_1, A_1, \cdots$ we combine into products E_1A_1, E_2A_2, $\cdots$ of the algebraic kind, since the original products were algebraic just as the product xy. The new expression for the sum of such factors we shall call a *potenzform of the second order* and denote it by the symbol $A^{(2)}$ so that

$$A^{(2)} = \sum_1^3 E_iA_i = a_{11}E_1^2 + a_{22}E_2^2 + a_{33}E_3^2 + 2a_{23}E_3E_2 + 2a_{31}E_3E_1 + 2a_{21}E_2E_1,$$

and this potenzform shall henceforth mean precisely what is expressed by the equation $A_2x^2=0$, that is, a curve of the second order. The quantities $E_1, {}^2E_2, {}^2E_3, {}^2E_2E_3, E_3E_1, E_1E_2$ are *the six units of the second order*. Conversely, the lückenform may easily be constructed, the potenzform being given. Moreover, if we want the curve itself from the potenzform, all we have to do is to substitute for the fundamental stäbe E_1, E_2, E_3 the coordinates ξ_1, ξ_2, ξ_3. Thus the potenzform $a_{23}E_2E_3+a_{31}E_3E_1+a_{12}E_1E_2$ represents the curve of the second order $a_{23}\xi_1\xi_3+a_{31}\xi_3\xi_1+a_{12}\xi_1\xi_2=0$ which is circumscribed about the fundamental triangle; the lückenform is

$$a_{23}[lE_2][lE_3] + a_{31}[lE_3][lE_1] + a_{12}[lE_1][lE_2].$$

In the dual case, starting with a polarity of the second class, we obtain a ternary lückenform of the second class

$$A_2 = [e_1L][a_1L] + [e_2L][a_2L] + [e_3L][a_3L] = \sum [e_iL_i][a_iL]$$

or, introducing the values $a_i = \sum A_{ik}e_k$, $A_{ik} = A_{kl}$,

$$A_2 = A_{11}[e_1L]^2 + A_{22}[e_2L]^2 + A_{33}[e_3L]^2 + 2A_{23}[e_2L][e_2L] + 2A_{31}[e_3L][e_1L] + 2A_{12}[e_2L][e_1L]$$

and the polar curve of the polarity P has the equation $a_2U^2 = 0$. Two lines U and V are conjugate when the bilinear form $a_2UV = 0$. The potenzform corresponding to the lückenform of the second class is now

$$A^{(2)} = A_{11}e_1^2 + A_{22}e_2^2 + A_{33}e_3^2 + 2A_{23}e_2e_2 + 2A_{31}e_3e_1 + 2A_{12}e_1e_2,$$

which represents the curve $a_2U^2 = 0$. The most general potenzform of the second class is derived from the *six units of the second class*, viz.: e_1^2, e_2^2, e_3^2, e_2e_3, e_3e_1, e_1e_2, and the polar curve may be obtained from it by substituting e_i for u_i in the equation of a conic of the second class.

The combinatory product of a potenzform of the second order and a potenzform of the second class is now defined as follows: $[A^{(2)}b^{(2)}] = A_2b^{(2)}$ and $[b^{(2)}A^{(2)}] = b_2A^{(2)}$, that is, we put the combinatory product equal to the product obtained by substituting for the first potenzform the corresponding lückenform and consider the extensive factor of the second potenzform as "filling in" factor intended for the lacunae of the lückenform. The following theorem is then proved: *The combinatory product* $[A^{(2)}b^{(2)}]$ *vanishes when, and only when, the two potenzformen are apolar.* There is however a "fly in the ointment" right here. This elaborate preparation of an algorithm to express the apolarity of two polar systems is not applicable to non-involutary or general reciprocities. We have to go back to the former condition $[rS] = 0$, equation (4).

In the remainder of the chapter special cases of apolarity are given and two theorems of von Staudt on polar triangles and quadrilaterals are proved.

In our journey through 950 pages of this treatise of 1250 pages we have met with the cubic only once, the cubic of involution mentioned above. It is therefore a pleasant relief when the author in the following Chapters XII and XIII allows us to renew our old acquaintances, the curves of the third order and class, although in a stranger garb. A lückenform of the third order is written

$$A_3 = a_{111}[lE_1]^3 + \cdots + 3a_{112}[lE_1]^2[lE_2] + \cdots + a_{123}[lE_2][lE_3][lE_3] + \cdots,$$

and a lückenform of the third class

$$a_3 = A_{111}[e_1L]^3 + \cdots + 3A_{112}[e_1L]^2[e_2L] + \cdots + 3A_{122}[e_1L][e_2L][e_2L] + \cdots,$$

the corresponding curves of the third order and the third class being

$$A_3x^3 = \{\sum a_{ikl}[lE_i][lE_k][lE_l]\}x^3 = 0,$$
$$a_3U^3 = \{\sum A_{ikl}[e_iL][e_kL][e_lL]\}U^3 = 0.$$

Several familiar theorems on polars of a cubic are then proved. The Hessian of a cubic is expressed by the vanishing of a combinatory product of three lacunary forms $H_3z^3=[A_3ze_1\cdot A_3ze_2\cdot A_3ze_3]$, where $H_3=[A_3le_1\cdot A_3le_2\cdot A_3le_3]$ is a ternary lückenform of the third order. The Hessian of a curve of the third class is represented as a combinatory product of three lückenformen of the third class in the form $h_3W^3=[a_3WE_1\cdot a_3WE_2\cdot a_3WE_3]=0$. The Grassmann point-calculus seems to be well adapted to a careful treatment of the inflectional tangents to a cubic, and the author takes special care in his discussion of the real and imaginary configurations connected with these tangents. The same care is also taken with the dual configuration of the 9 cuspidal tangents to the cubic of the third class.

Chapter XIV deals with nets of conics and their duals, webs of conics. The chapter ends with a section on domains of polarity, and a final section on the Cayleyan curve of a net (web). In the last two chapters metrical properties of conics and their relation to the circular points at infinity are considered. Oblique and rectangular axes are used in turn, and a host of special theorems are produced. The projective properties of some special nets and webs of conics are given due consideration in the last chapter.

As a magisterial work the Projective Geometry of H. Grassmann, Jr. takes a high rank. Although somewhat diffuse at times, it is a marvel of painstaking care and exactitude. It is singularly free from missprints. On page 145, (vol. 2, Part 2) sixth line from bottom, the words "der Ableitszahlen" should be inserted after the word "Funktionen." On p. 284, line 19 from above, the word "und 789" should be inserted after the number "791." On p. 346, last line, "Potenzreihen" should read "Potenzformen."

It is to be regretted that the author had to depart before he could accomplish what was doubtless his intention, an extension of the work to the quaternary field. The notes published by the author in Volume I of Grassmann's Gesammelte Werke, pages 438–464, seem to be a tentative beginning in this direction.

"Hans bok kom ikke, han selv gik derhen,
Hvor tankernes lov ikke skrives med pen."
—B. Björnson

JOHN EIESLAND

THE NEW RIEMANN-WEBER—VOLUME II

Die Differential- und Integralgleichungen der Mechanik und Physik. Zweiter, physikalischer Teil, edited by P. Frank with the assistance of H. Faxén, R. Fürth, Th. v. Kármán, Fr. Noether, C. W. Oseen, A. Sommerfeld, E. Trefftz. Braunschweig, Vieweg, 1927. 863 pp.+88 figures. Paper, 53 Rentenmark; bound, 58 Rentenmark.

The first part, or mathematical section of this work, was reviewed by the present writer in the May-June 1927 number of this Bulletin. It outlines in a very broad manner the mathematical tool-box of the theoretical physicist of the present day, which covers nearly the whole realm of analysis and large portions of the other departments of mathematics.

For the second volume, the goal adopted by the editors was to accomplish for the last Riemann-Weber volume of 1910, that which Weber did in 1900, for Riemann's original *Partielle Differentialgleichungen der mathematischen Physik*. We may quote freely from the preface to the effect that in spite of fad and fancy in physical hypotheses, the mathematical formulations and methods are relatively constant. Examples are cited in the application of the planetary perturbation theory of Laplace and Lagrange to atomic states of energy; in the identity between the equations for the transfer of heat due to Fourier and that of the phenomena of diffusion in liquids; in the universality of the wave equations of D'Alembert, Euler and Gauss in sound, light, and atomistics; in the appearance of classical potential theory of Laplace, Gauss, Cauchy and Riemann in the modern theories of lift and drag on an airplane wing. Contrarywise, the fluid theories of heat and electricity are in the discard, atoms, molecules and electrons may even follow, Newton's hypothesis of Universal Gravitation may go out of fashion, but the fundamental analytic framework in terms of which the corresponding phenomena are described is relatively permanent. More recently, the influence of the mathematical frame of reference on hypothesis has been considerable.

The second volume of the present work joins on to the first in very much the same manner as superstructure to foundation. Despite the fact that volume two is the work of seven writers, in each case different from each of the eight collaborators on the first, a unified picture of the entire field wherein differential and integral equations function in the field of mathematical physics has been secured. There is no discussion of merit or lack of merit of physical hypothesis, very little comparison between mathematical result and experimental observation, but for the sake of unity the entire purpose is concentrated on the development of those particular analytic formulations in the realm of physical phenomena, particularly to recent developments such as quanta, Brownian movements, radiation, etc., together with technical applications of special importance such as aeronautics and radio. Some of this material on these topics appears in print for the first time.

The following is a summary by chapter headings:

Part I (by P. Frank). Analytic mechanics. 1. The differential equations of general mechanical systems. 2. Stability and small oscillations. 3. The equations of motion of rigid bodies. 4. Methods in perturbation theory. 5. Problems of celestial and atomic mechanics.

Part II (by R. Fürth). Conduction of heat and diffusion. 6. Free conduction of heat and diffusion. 7. Induced conduction of heat and diffusion.

Part III (by Fr. Noether). The stationary and quasi-stationary electromagnetic fields. 8. Electrostatics. 9. Steady electric currents. 10. Magnetostatics. 11. Quasi-stationary currents and waves.

Part IV (by A. Sommerfeld). Electromagnetic oscillations. 12. General theorems and methods. 13. Theory of refraction. 14. Resistance and skin effect. 15. Waves on wires. 16. Wireless telegraphy.

Part V (by E. Trefftz). Mechanics of continua. 17. Mathematical foundations of the theory of elasticity. 18. Problems in elastic equilibrium. 19. Dynamical problems of elastic theory. 20. Ideal fluids (by Th. v. Kármán). 21. Motion of viscous fluids (by Faxén and Oseen).

Part I is especially complete on the general equations of mechanics and their solution. The treatment is entirely classical and prepares the ground for the problems of classical atomic mechanics. Relativistic mechanics is not included. The general equations of motion and stability of an airplane are treated. Part II on heat and diffusion follows in the main the classical treatment, except that the problems of diffusion are treated in parallel with those of heat flow. Part III on the static and quasi-static electromagnetic fields covers both the classic and the modern portions of the subject. The same may be said of electromagnetic oscillations. Here, the emphasis is on the theory of radiation and electric wave phenomena with applications to radio-activity, light and wireless. Finally, Part IV treats the mechanics of continuous media such as elasticity, motion of fluids, both perfect and viscous, and related problems. The last two chapters of this section contain new material of importance in aerodynamics.

It is a fairly simple matter to point out omissions in the writing of as comprehensive a work as the present one. There is the immediate reply to this of the authors of a desire to avoid expansion into encyclopedic proportions. However, it does seem that it would have been in order to include some applications of difference equations or algebraic systems related to differential and integral equations. One might also venture to suggest the inclusion of some chapters on the newer geometry and the theory of geodesics with applications to problems in gravitation, elasticity, etc.

Though definitely outside of the program of the present two volumes, if one were to add in an appendix, chapters on statistics and probability in relation to states of energy, and group and number theory in relation to the concept of symmetries in crystal structure, one would have a truly representative and comprehensive picture of the interlacing of mathematical reference frames and physical hypotheses in the description of the natural phenomena of the universe.

H. J. ETTLINGER

SHORTER NOTICES

Theorie der Kristallstruktur. By Arthur Schoenflies. Berlin, Gëbrüder Borntraeger, 1923.

This book is essentially a new edition of Professor Schoenflies' *Krystallsysteme und Krystallstruktur* which appeared in 1891. Like its predecessor its primary concern is the deduction of the 230 crystallographically significant space groups. Though remaining purely geometric in the details of its reasoning, this deduction has been so rewritten as to make it somewhat shorter and more concise. The insertion of numerous figures is a great help towards the picturing of individual groups. As before these groups are described through the statement of both their sub-groups and the coordinates of equivalent positions within them. This description has, however, been amplified and improved through a listing of the symmetry properties associated with points lying in elements of symmetry.

Since the discovery of X-ray diffraction the theory of space groups has become of immediate and every day use in experimental physics. The realization of this changed importance has inevitably influenced Professor Schoenflies' treatment and has led him to lay greater stress upon items of applied crystallographic interest. A chapter has accordingly been inserted which outlines the practical usefulness of space groups in studies of the positions of atoms in crystals. Although this account will not meet the needs of the practicing crystal analyst, Professor Schoenflies' books remain the only suitable source of information for those interested in the derivation of space groups.

R. W. G. WYCKOFF

Mathematics in Liberal Education. By F. Cajori. Boston, Christopher, 1928.

Do you believe in the "transfer of training"? Do you think that the study of mathematics yields a certain discipline which can be applied to situations outside of mathematics—which can be "transferred"? If you do, psychologists today would mostly agree with you.

But when they proceed by standard tests and other devices to measure the amount of this transfer, their findings are meager or nil. Professor Cajori doubts the psychologists' methods and suggests that in the absence of adequate objective measures it is not unscientific to consider subjective measures such as the opinions of philosophers and men of science.

His little book presents the recorded opinions of leading thinkers of all ages on the disciplinary value of mathematics. These quotations are arranged in chronological order, and include all opinions *contra* as well as *pro*. Which being counted, it appears that the *pro*'s have it.

This is a book for the believer, for the regeneration of faith in the faithful. And for him who battles daily with those who value mathematics only as a tool, or who sit in the seat of the scornful, it increaseth strength.

RALPH BEATLEY

Gewöhnliche Differentialgleichungen. By Dr. G. Hoheisel. Sammlung Göschen. Berlin and Leipzig, Walter de Gruyter, 1926. 159 pp.

This little book contains a surprising amount of material. Beginning with the very elements of the subject, it is remarkably comprehensive.

As a text-book for the American or English student, it will not be found entirely satisfactory. The treatment is not very systematic, nor is the usual elementary ground completely covered. And there are no exercises for the student. But for the enquiring student, and as collateral reading this little book will undoubtedly be found very attractive.

The author presents relatively simply some fairly difficult topics. While he does not go deeply into them, he introduces quite skillfully such subjects as behavior of integral curves in the neighborhood of singular points, integration by means of infinite series and definite integrals, and boundary value problems.

A. COHEN

Theorie der Differentialgleichungen. Vorlesungen aus dem Gesamtgebiet der gewöhnlichen und der partiellen Differentialgleichungen. Zweite neubearbeitete Auflage.* By Ludvig Bieberbach. Berlin, Julius Springer, 1926. ix+358 pp.

As in the original edition, the author has set himself the difficult task of carrying the reader from the very elements of the subject to the comprehension of some of the recent developments. In his preface he recognizes the impossibility of covering the whole field. So he endeavors to make a selection of topics. That he succeeds in making a most attractive one there can be no doubt. But skillful as the author is in masterful presentation, it seems almost an impossible undertaking to carry out the aims of the book in so few pages.

The second edition follows the general plan of the earlier one. There are many minor changes and additional cuts. Several chapters are practically rewritten. Although larger by some fifty pages, here too some of the attempts to give a hasty survey of advanced theories seem less readable (because of their brevity) than the original memoirs.

But the book a pioneer of its kind in the subject of differential equations, is a valuable addition to the literature, and serves excellently as a guide towards some very interesting and important fields of development.

A. COHEN

* For a review of the first edition see this Bulletin, vol. 31 (1925), p. 279.

NOTES

The concluding number of volume 30 of the Transactions of this Society (October, 1928) contains the following papers: *Some non-linear problems in approximation*, by D. Jackson; *Second-order differential systems with integral and k-point boundary conditions*, by W. M. Whyburn; *Analytic functions of hypercomplex variables*, by P. W. Ketchum; *Concerning limiting sets in abstract spaces*, by R. G. Lubben; *Riesz summability for double series*, by F. M. Mears; *The inverse problem of the calculus of variations in higher space*, by D. R. Davis; *Types of motion of the gyroscope*, by A. H. Copeland; *The invariant integral and the inverse problem in the calculus of variations*, by T. H. Rawles; *The projective differential geometry of systems of linear homogeneous differential equations of the first order*, by E. P. Lane; *The behavior of a boundary value problem as the interval becomes infinite*, by W. E. Milne; *The second derivative of a polygenic function*, by E. Kasner; *Diophantine equations in division algebras*, by R. G. Archibald; *On the degree of approximation to an analytic function by means of rational functions*, by J. L. Walsh; *Existence and oscillation theorems for non-linear differential systems of the second order*, by W. M. Whyburn. The Index of volumes 21–30 (1920–28) is bound with this number.

The concluding number of volume 50 of the American Journal of Mathematics (October, 1928) contains: *Affinely connected function space manifolds*, by A. D. Michal; *A theorem concerning the affine connection*, by T. Y. Thomas; *Concerning subsets of a continuous curve which can be connected through the complement of the continuous curve*, by W. L. Ayres; *Admissible numbers in the theory of probability*, by A. H. Copeland; *The differential invariants of inversive geometry*, by B. C. Patterson; *Expansions of the Neumann type in terms of products of Bessel functions*, by G. Stevenson; *Canonical forms for ordinary homogeneous linear differential equations of the second order with periodic coefficients*, by E. Swift; *Periodic orbits of three finite masses about the equilateral triangle points*, by H. E. Buchanan; *The problem of plane involutions of order $t>2$*, by F. R. Sharpe. A portrait of Goursat is bound with this number.

Professor G. D. Birkhoff of Harvard University delivered a lecture on *The mathematical basis of art* at the Ohio State University on December 3, 1928.

Professor E. R. Hedrick, Editor-in-Chief of this Bulletin, visited and lectured at the following institutions during November and December, 1928: University of Missouri, November 26; Ohio State University, November 27–28; University of Cincinnati, November 30–December 3; University of Kentucky, December 4; Oberlin College, December 5–6; University of Michigan, December 7; University of Wisconsin, December 10; Northwestern University, December 11; University of Indiana, De-

cember 12; University of Pittsburgh, December 13–17; University of West Virginia, December 18; University of Pennsylvania, December 19.

A new journal entitled Boletín del Seminario Mathemático Argentino has been founded, under the editorship of Professor J. Rey Pastor. The first number appeared in September, 1928.

The Société Mathématique Suisse has founded a new mathematical journal, the Comentarii Mathematici Helvetici; Professor Andreas Speiser, of the University of Zurich, is editor-in-chief. The journal is published by Orell-Füssli, of Zurich.

On the occasion of the meeting of the British Association for the Advancement of Science at Glasgow, the University of Glasgow conferred honorary doctorates on Sir William Bragg, president of the Association, and Professor P. Zeeman, of the University of Amsterdam, among others. The Association will hold its summer meeting of 1929 in South Africa, at Cape Town, Johannesburg, and Pretoria.

At the thirteenth annual meeting of the Optical Society of America, held in Washington, October 31 to November 3, 1928, under the joint auspices of that society and the Bureau of Standards, was celebrated the semicentennial of the publication of Professor A. A. Michelson's first communication on the velocity of light. On this occasion, Professor Michelson and his collaborators, Messrs. Pease and Pearson, delivered an address on *Results of repetition of the Michelson-Morley experiment.*

The professors and students of the Faculty of Sciences of the University of Turin have placed a commemorative medallion in honor of the late Professor Corrado Segre in one of the rooms of the Mathematical Institute of the University.

The Sacchetti Prize designed to acknowledge eminent scientific work of the teaching staff of the University of Bologna has been awarded to Professor S. Pincherle, who has retired.

Sir Ernest Rutherford has been elected a foreign honorary member and Professor Erwin Schrödinger a foreign corresponding member of the Vienna Academy of Sciences.

Professor Raymond Pearl, of Johns Hopkins University, has been elected president of the International Union for the Study of Demographic Problems.

The following have been appointed to National Research Fellowships in mathematics for 1928–29: A. A. Albert, W. L. Ayres, F. R. Bamforth, Theodore Bennett, Alonzo Church, T. F. Cope, C. M. Cramlet, Jesse Douglas, D. A. Flanders, J. J. Gergen, M. S. Knebelman, Lincoln LaPaz, Morris Marden, T. W. Moore, Hillel Poritsky, N. E. Rutt, I. M. Sheffer, Marie J. Weiss. Messrs. Ayres, Church, and Douglas hold fellowships supported by the International Education Board.

Professor L. Fantappié, of the University of Cagliari, has been transferred to the chair of algebraic analysis in the University of Palermo.

Associate Professor J. Lense, of the Munich Technical School, has been promoted to a full professorship of mathematics.

Professor B. Levi, of the University of Parma, has been transferred to the chair of theory of functions in the University of Bologna.

Dr. G. Mammana has been appointed associate professor of calculus in the University of Cagliari.

Associate Professor R. Ortvay, of the University of Szeged, has been appointed associate professor of theoretical physics at the University of Budapest.

Professor Johann Radon, of the University of Erlangen, has been appointed professor of mathematics at the University of Breslau.

Dr. C. Weber has been appointed professor of mechanics at the Dresden Technical School.

Professor Gregor Wenzel, of the University of Leipzig, has been appointed professor of theoretical physics at the University of Zurich.

Mr. P. T. Copp has been appointed head of the department of mathematics at Valparaiso University.

Dr. Mary G. Haseman has been appointed professor of mathematics at Hartwick College, Oneonta, N. Y.

Associate Professor L. S. Johnston, of Pennsylvania State College, has been appointed head of the department of engineering mathematics at the University of Detroit.

Assistant Professor J. R. Kline has been promoted to a full professorship at the University of Pennsylvania.

Dr. O. J. Peterson has been appointed head of the department of mathematics at the State Teachers' College at Emporia, Kansas.

Mr. M. R. Richardson has been appointed assistant professor of mathematics at the Georgia School of Technology.

Dr. W. C. Risselman has been appointed assistant professor of mathematics at the University of Pittsburgh.

Assistant Professor J. B. Scarborough, of the United States Naval Academy, has been promoted to an associate professorship of mathematics.

During the second half of the present academic year, beginning about February 1, Professor H. W. Tyler, of the Massachusetts Institute of Technology, will be on leave of absence in Washington, D.C., in order to establish there a permanent office of the American Association of University Professors, of which he has acted as Secretary for many years.

Dr. John Williamson has been appointed associate in mathematics at Johns Hopkins University.

The following appointments to instructorships in mathematics are announced:

George Washington University, Mr. E. W. Woolard of the United States Weather Bureau;

University of Michigan, Mr. P. M. Swingle;

Northwestern University, Mr. R. H. Bardell;

Pennsylvania State College, Messrs. L. T. Dunlap, W. O. Gordon, J. A. Hamilton, P. V. Kunkel, and R. E. Peterson;

St. John's College, Mr. G. C. Vedova.

Professor Richard Birkeland, of the University of Oslo, died April 10, 1928, at the age of forty-nine.

Professor Victor Eberhard, of the University of Halle, died April 28, 1928, at the age of sixty-seven.

Mr. Jun Hirai, of the Shizuoka High School, has died.

Professor O. Henrici, retired, of the London Technical College, has died, at the age of eighty-eight.

Professor E. Wiechert, of the University of Göttingen, has died at the age of fifty-seven.

Dr. Wilhelm Wien, professor of physics at the University of Munich, died August 30, 1928, at the age of sixty-five.

Professor R. Sterneck, of the University of Graz, has died, at the age of fifty-seven.

Professor H. B. Fine, of Princeton University, was fatally injured by an automobile on the evening of Friday, December 21, and died about 1 A.M. on Saturday, December 22, 1928, at the age of seventy. He had been a member of the American Mathematical Society since 1891, and was president of the Society during the years 1911–1912.

Miss Marion Reilly, of Bryn Mawr College, died in January, 1928. Miss Reilly had been a member of the American Mathematical Society since 1913.

Professor C. D. Rice, of the University of Texas, died May 28, 1928, at the age of sixty-five. Professor Rice had been a member of the American Mathematical Society since 1920.

Professor Alexander Ziwet, of the University of Michigan, died November 18, 1928. He had been a member of the American Mathematical Society since 1891, and had served for many years as a member of the board of editors of this Bulletin.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

ARISTOTLE. The works of Aristotle, translated into English under the editorship of W. D. Ross. Volume 1. London, Oxford University Press, 1928. 4+652 pp.

VAN DEN BERG (P. J.). Die scherpe hoek. Velp, 1928. 22+25+22 pp.

BERNAYS (P.). See HILBERT (D.).

BIANCHI (L.). Congruenze di sfere di Ribaucour e superficie di Peterson. Bologna, Zanichelli, 1928. 103 pp.

BIERNACKI (M.). Sur les équations algébriques contenant des paramètres arbitraires. (Thèse, Paris.) Cracovie, Imprimerie de l'Université, 1928. 145 pp.

BLANC (E.). See PICKARD (E.).

BORTOLOTTI (E.). La scuola matematica di Bologna. Bologna, Zanichelli, 1928. 75 pp.

BOULIGAND (G.). Leçons de géométrie analytique. Paris, Vuibert, 1928. 498 pp.

CAJORI (F.). A history of mathematical notations. Volume I: Notations in elementary mathematics. Chicago, Open Court, 1928. 16+451 pp. $6.00.

—— Mathematics in liberal education. Boston, Christopher, 1928. 169 pp.

CARTAN (E.). Leçons sur la géométrie des espaces de Riemann. Paris, Gauthier-Villars, 1928. 6+273 pp.

FARWICK (W.). See LEINEWEBER (N.).

HARDY (G. H.). A course of pure mathematics. Cambridge, University Press, 1928. 12+455 pp.

HARMEGNIES (R.). See JULIA (G.).

HILBERT (D.). Die Grundlagen der Mathematik. Mit Zusätzen von Hermann Weyl und Paul Bernays. (Hamburger Mathematischen Einzelschriften, Heft 5.) Leipzig, Teubner, 1928. 28 pp.

JULIA (G.). Exercices d'analyse rédigés par R. Harmegnies et R. Julia. Tome I. Paris, Gauthier-Villars, 1928. 8+454 pp.

JULIA (R.). See JULIA (G.).

JUNGE (G.). Einführung in Wesen und Wert der Mathematik. Karlsruhe, G. Braun, 1928. 92 pp.

KNOPP (K.). Theory and application of infinite series. Translated from the second German edition by R. C. Young. London, Blackie, 1928. 12+571 pp.

LEBESGUE (H.). Leçons sur l'intégration et la recherche des fonctions primitives. 2e édition. Paris, Gauthier-Villars, 1928. 16+342 pp.

LEINEWEBER (N.) und FARWICK (W.). Wertetafel. 2te Auflage. Münster, Aschendorffsche Verlagsbuchhandlung, 1927. 8+112 pp.

LIEDLOFF (W.). Beiträge zur Psychologie der mathematischen Schulbegabung. Langensalza, Verlag Beltz, 1928. 94 pp.

LIPPS (H.). Untersuchungen zur Phänomenologie der Erkenntnis. Teil 2: Aussage und Urteile. Bonn, Friedrich Cohen, 1928. 105 pp.

MENGER (K.). Dimensionstheorie. Leipzig, Teubner, 1928. 4+320 pp.

VON MISES (R.). Wahrscheinlichkeit, Statistik und Wahrheit. Berlin, Springer, 1928. 98 pp.

PADALINO (F.). Five sheets of diagrams illustrating the Delian problem, rectification of arches, and trisection of angles. New York, 1928.

PHILLIPS (H. B.). Calculus. London, Chapman and Hall, 1927. 353 pp.

PICARD (E.). Leçons sur les équations fonctionnelles avec des applications à divers problèmes d'analyse et de physique mathématique. Rédigées par E. Blanc. Paris, Gauthier-Villars, 1928. 187 pp.

—— Selecta. Paris, Gauthier-Villars, 1928. 8+304 pp.

PRASAD (G.). Six lectures on recent researches in the theory of Fourier series. Calcutta, University of Calcutta, 1928. 14+139 pp.

ROSS (W. D.). See ARISTOTLE.

ROWE (J. E.). Introductory mathematics. London, Pitman, 1928. 293 pp.

SIERPIŃSKI (W.). Leçons sur les nombres transfinis. (Borel Series.) Paris, Gauthier-Villars, 1928. 6+240 pp.

SYMON (A.). The new geometry. Part 2. Glasgow, Gibson, 1928.

TURNBULL (H. W.). The theory of determinants, matrices, and invariants. London, Blackie, 1928. 16+338 pp.

WERKMEISTER (P.). Einführung in die Ausgleichungsrechnung nach der Methode der kleinsten Quadrate. Stuttgart, Konrad Wittwer, 1928. 6+198 pp.

WEYL (R.). See HILBERT (D.).

YOUNG (R. C.). See KNOPP (K.).

PART II. APPLIED MATHEMATICS

ADAMI (F.). Die Elektrizität. 4te Auflage. Teil 2. Leipzig, Reclam Verlag, 1928. 180 pp.

AUERBACH (F.) und HORT (W.). Handbuch der physikalischen und technischen Mechanik. Band II, 1te Lieferung; Band V, 1te und 2te Lieferung; Band VI, 1te Lieferung; Band VII, 1te Lieferung. Leipzig, Barth, 1927–28. 8+404+9+274+6+718+8+460+4+238 pp.

BASU (K.) and KARMAKAR (S.). An intermediate course of statics. Allahabad, Indian Press, 1928. 18+308 pp.

BAUR (F.). Korrelationsrechnung. (Mathematisch-Physikalische Bibliothek.) Leipzig, Teubner, 1928. 6+57 pp.

BAVINK (B.). Die Hauptfragen der heutigen Naturphilosophie. I. Berlin, Salle, 1928. 121 pp.

BERTHOUD (A.). Photochimie. Paris, Doin, 1928. 8+323 pp.

BEYER (K.). See MOHR (O.).

BIRD (G. W.). Examples in the strength and elasticity of materials. London, Arnold, 1927.

BOTHE (W.). See KOHLRAUSCH (F.).

BOWLEY (A. L.). Edgeworth's contributions to mathematical statistics. London, Royal Statistical Society, 1928. 7+139 pp.

BRILL (A.). Vorlesungen über allgemeine Mechanik. München, Oldenbourg, 1928. 8+356 pp.

BRILLOUIN (L.). See DE BROGLIE (L.).

BRODHUN (E.). See KOHLRAUSCH (F.).

DE BROGLIE (L.) and BRILLOUIN (L.). Selected papers on wave mechanics. London, Blackie, 1928. 151 pp.

CANDY (H. C. H.). A manual of physics, theoretical and practical, for medical students. London, Cassell, 1928. 496 pp.

CONGRÈS INTERNATIONAL POUR L'ESSAI DES MATÉRIAUX. Proceedings of the International Congress for Testing Materials, Amsterdam, 1927. 2 volumes. The Hague, Nijhoff, 1928. 20+581+12+737 pp.

COPPEL (T.), FOURNIER (G.), et YOVANITCH (—.). Quelques suggestions concernant la matière et le rayonnement. Paris, Blanchard, 1928. 46 pp.

COVENTRY, (W. B.). The mechanics of rowing. London, Spon, 1928. 8+70 pp.

CREIGHTON (H. J.). Principles and applications of electro-chemistry. Volume 1: Principles. New York, Wiley, 1928. 16+488 pp.

CZWALINA (A.). See KLEOMEDES.

DANNEEL (H.). Elektrochemie und ihre physikalisch-chemischen Grundlagen. Band 4: Elektrolyse. (Sammlung Göschen.) Berlin, de Gruyter, 1928. 144 pp.

FLECHSENHAAR (A.). Einführung in die Finanzmathematik. 2te Auflage, bearbeitet in Verbindung mit F. Fleege-Althoff. Leipzig, Teubner, 1927. 6+109 pp.

FLEEGE-ALTHOFF (F.). See FLECHSENHAAR (A.).

FORSYTH (C. H.). Introduction to the mathematical theory of finance. New York, Wiley, 1928. 5+205 pp.

FORTRAT (R.). Introduction à l'étude de la physique théorique. Fascicule 3: Thermodynamique. Paris, Hermann, 1927. 189 pp.

FOTHERINGHAM (J. K.). See LANGDON (S.).

FOURNIER (G.). See COPPEL (T.).

GIBSON (C. R.). Modern conceptions of electricity. London, Seeley, Service and Company, 1928. 284 pp.

GIEBE (E.). See KOHLRAUSCH (F.).

GRAETZ (L.). Die Elektrizität und ihre Anwendungen. 23te wesentlich neugearbeitete Auflage. Stuttgart, Engelhorn, 1928. 16+818 pp.

GROTRIAN (W.). Graphische Darstellung der Spektren von Atomen und Ionen mit ein, zwei und drei Valenzelektronen. 1ter and 2ter Teil. (Struktur der Materie in Einzeldarstellungen, Band 7.) Berlin, Springer, 1928. 13+245+10+168 pp.

GRÜNEISEN (E.). See KOHLRAUSCH (F.).

VON HAHN (F. V.). Dispersoidanalyse. Die Methoden der Teilchengrössenbestimmung und ihre theoretischen Grundlagen. Dresden, Steinkopf, 1928. 24+553 pp.

HARWOOD (P. J.). A theory of the solar system. Part 2. Brighton, the Author, 1928. 2+64 pp.

HOLBORN (L.). See KOHLRAUSCH (F.).

HORT (W.). See AUERBACH (F.).

JEANS (J. H.). Eos, or the wider aspects of cosmogony. London, Kegan Paul, and New York, Dutton, 1928.

JELLINEK (K.). Lehrbuch der physikalischen Chemie. 2te, vollständig umgearbeitete Auflage. Band 1, und Band 2, Lieferung 4. Stuttgart, Enke, 1928. 54+966+272 pp.

JOHNSON (R. C.). Spectra. London, Methuen, 1928. 7+104 pp.

KARMAKAR (S.). See BASU (K.).

KENNELLY (A. K.). Vestiges of pre-metric weights and measures persisting in metric-system Europe. New York, Macmillan, 1928. 13+189 pp.

KLEOMEDES. Die Kreisbewegung der Gestirne. Übersetzt und erläutert von A. Czwalina. (Ostwalds Klassiker der Exakten Wissenschaften.) Leipzig, Akademische Verlagsgesellschaft, 1927. 93 pp.

KOHLRAUSCH (F.). Lehrbuch der praktischen Physik. 15te, stark vermehrte Auflage. Neu bearbeitet von W. Bothe, E. Brodhun, E. Giebe, E. Grüneisen, L. Holborn, K. Scheel, und O. Schönrock. Leipzig, Teubner, 1927. 30+832 pp.

KRÜGER (F.). Materie und Energie im Welt-Geschehen. Greifswald, Verlag Bamberg, 1928. 29 pp.

LANGDON, (S.) and FOTHERINGHAM (J. K.). The Venus tablets of Ammizaduga. A solution of Babylonian chronology by means of the Venus observations of the first dynasty. With tables for the computation by C. Schoch. London, Oxford University Press, 1928. 6+109+16 pp.

MAHLER (G.). Physikalische Formelsammlung. 5te, völlig umgearbeitete Auflage, besorgt von K. Mahler. Berlin, de Gruyter, 1927. 162 pp.

MAHLER (K.). See MAHLER (G.).

MARTIN, (G.). A treatise on chemical engineering applied to the flow of industrial gases, steam, water and liquid chemicals, including the pneumatic transport of powders and granulated materials. London, Crosby and Lockwood, 1928. 23+242+19 pp.

MOELLER (H.). Statistik. Berlin, Spaeth und Linde, 1928. 14+149 pp.

MOHR (O.). Abhandlungen aus dem Gebiete der technischen Mechanik. 3te erweiterte Auflage zur Jahrhundertfeier der Technischen Hochschule Dresden. Herausgegeben von K. Beyer und H. Spangenberg. Berlin, Wilhelm Ernst, 1928. 15+622 pp.

MOTTA (M.). La costituzione e l'evluzioni della materia secondo gli ultimi risultati della fisico-chimica. Roma, Loescher, 1928.

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Many others are in press or in preparation.

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OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

NEW YORK CITY, February 23, 1929.

Abstracts must be in the hands of Professor R. G. D. Richardson, Secretary of the Society, 501 West 116th St., New York City, not later than February 2. At the invitation of the program committee, Professor J. L. Walsh will deliver an address entitled *The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions.*

NEW YORK CITY, March 29-30, 1929.

Abstracts must be in the hands of Professor R. G. D. Richardson, Secretary of the Society, 501 West 116th St., New York City, not later than March 8. At the invitation of the program committee, Professors G. H. Hardy and H. Weyl will present addresses on *Modern work in the theory of ordinary trigonometric series*, and on *Fourier series and almost periodic functions from the standpoint of the theory of groups*, respectively.

CHICAGO, March 29-30, 1929.

Abstracts must be in the hands of Professor M. H. Ingraham, Associate Secretary of the Society, University of Wisconsin, Madison, Wis., not later than March 8. At the invitation of the program committee, Professor E. B. van Vleck will deliver a symposium lecture on *The location of the roots of polynomials and entire functions.*

STANFORD UNIVERSITY, California, April 6, 1929.

Abstracts must be in the hands of Professor B. A. Bernstein, University of California, Berkeley, Calif., not later than March 5.

R. G. D. RICHARDSON,
Secretary of the Society.

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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WHOLE NUMBER 358

CONTENTS

	PAGE
The October Meeting in New York. By ARNOLD DRESDEN	1
The Fifty-Sixth Regular Meeting of the San Francisco Section. By B. A. BERNSTEIN	11
The Amherst Colloquium. By H. S. WHITE	16
The Heroic Age of Geometry. By J. L. COOLIDGE	19
The Critical Points of Functions and the Calculus of Variations in the Large. By MARSTON MORSE	38
The Forms $ax^2+by^2+cz^2$ which Represent All Integers. By L. E. DICKSON	55
The Theory of Multiple Implication and its Application to the Generalized Problem of Epemenides. By H. B. SMITH	60
Concerning Covariants of the Rational Plane Quartic Curve with Compound Singularities. By J. H. NEELLEY	67
On the Removal of Four Terms from an Equation by Means of a Tschirnhaus Transformation. By RAYMOND GARVER	73
Some Properties of Multi-Coherent Continua. By W. A. WILSON	78
Concerning Collections of Cuttings of Connected Point Sets. By G. T. Whyburn	87
Approximate Solutions of Certain General Types of Boundary Problems from the Standpoint of Integral Equations. By R. A. WHELAN	105
Linnaeus Wayland Dowling—In Memoriam	123
Grassmann's Projective Geometry, Volume II. By JOHN EIESLAND	124
The New Riemann-Weber, Volume II. By H. J. ETTLINGER	133
Shorter Notices	135
Notes	137
New Publications	141

For official Communications and Notices, see the inside of the back cover.